PROCUREMENT POLICY AND ALLOCATION POLICY OF COMMON COMPONENT IN A TWO-ECHELON ASSEMBLE-TO-STOCK SYSTEM

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CONTENTS

SUMMARY

This dissertation addresses the procurement and component allocation policies in the presence of component-sharing, or component commonality, in a two-echelon Assemble-To-Stock (ATS) manufacturing system, where an end-product is Assemble from several common components based on the demand forecast. This research work is conducted in three phases.

Firstly, we use a probabilistic model to study the component-sharing effects by comparing a particular component-sharing policy, namely the equal fractile allocation policy, with another policy that does not allow component-sharing. The equal fractile allocation policy allocates components such that all products will have equal probability of running out of stock after the allocation. The probabilistic analysis shows that when each type of component is shared by at least two end-products, the equal fractile allocation policy will always help to reduce the safety stock level required for high service standards. We also look at a special scenario of the equal fractile allocation policy with regard to inventory cost considerations. We show that this problem can be formulated as a newsvendor problem.

Secondly, we extend the allocation model to consider the total cost, and propose a component allocation policy that minimizes the total cost. This policy takes advantage of the sharing of common components. Simulation, Infinitesimal Perturbation Analysis (IPA) and steepest descent algorithm are used to find the order-up-to levels of components for the proposed allocation policy under uncorrelated and stationary demands. Its effectiveness is compared with two policies that do not allow the sharing

of common components. The results reveal that the proposed allocation policy always gives the lowest inventory cost due to the risk pooling of common components.

Thirdly, we formulate a two-stage problem to determine the optimal quantity to procure and the optimal quantity to allocate for the system with the auto-regressive demands are introduced. As the demands are correlated, there is value to the latest demand information so that we can more accurately forecast future demands as this information becomes available. We update the forecast demands at every review period and determine the order quantity and allocation quantity based on the latest demand information. We use the sample average approximation method to help in determining the optimal order quantity. The results reveal that the dynamic order-up-to level that considers component-sharing consistently outperforms the constant order-up-to level policy.

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NOTATIONS /NOMENCLATURE

AC	Average total cost
$AC_t(\vec{Y})$	Total cost at period t as a function of $\vec{Y}$
$a_{i,t}$	Allocation quantity for product *i* at period *t* (the nominal path)
$a'_{i,t}$	Allocation quantity for product *i* at period *t* for the perturbed path
$\vec{a}_t$	Vector of allocation quantities at period *t*
$\vec{a}_t^{(\omega_n)}$	Vector of allocation quantities at period *t* for sample path ω_n
$\Delta a'_{i,t}$	Change in the allocation quantity (due to the perturbation effect)
AR(P)	Auto-regressive process of order P
ATS	Assemble-To-Stock
$C_{t+l}\left(\vec{a}_t \mid \mathrm{X}_t, \vec{O}_{t-L}\right)$	Modified total cost at period *t+l* as a function of allocation quantities given system state at period *t* and order quantities at period *t-L*
$d_{i,t}$	Demand of product *i* at period *t* which has been realized
Δ	Normalized safety stock, K_2 - K_1
$D_{i,t}$	Demand of product *i* at period *t*
$\vec{e}_i$	The *i*th standard basis vector of vector space $\mathbf{R}^I$
$\vec{e}_j$	The *j*th standard basis vector of vector space $\mathbf{R}^J$
E_1	Event 1 occurs
G_{ij}	Quantity of component *j* required to make one unit of product *i*
$\vec{G}$	Product-component matrix or product-structure matrix
h_i	Holding cost per unit of product *i* on-hand per period

h'_i	Incremental holding cost per unit of product *i* on-hand per period
i	Index for product
I	Number of products
$I_{i,t}^{+}$	Amount of product *i* on-hand at the end of period *t*
$I_{i,t}^{-}$	Amount of backlogged demands for product *i* at the end of period *t*
IPA	Infinitesimal Perturbation Analysis
j	Index for component
J	Number of components
K	Safety factor
K_1	Safety factor for Equal Fractile Allocation Policy
K_2	Safety factor for Pure Push Policy
KKT	Karush Kuhn Tucker
l	Assembly lead time
L	Delivery lead time
L_j	Delivery lead time of component *j*
n	Number of iterations
N_ω	Number of sample paths selected
$O_{j,t}$	Order quantity of component *j* at period *t*
$o_{i,t}$	Order quantity of product *i* at period *t*
$\vec{O}_t$	Order quantity of component at period *t*
p_i	Penalty cost per unit of backlogged demand for product *i* per period

$R(Z_{i,t})$	Right-hand unit normal loss integral
$\vec{S}$	Vector of order-up-to levels of products
$s_{i,t}$	Inventory position of product *i*, consisting of work-in-process in the assembly process and the net inventory of product *i*, before the allocation at period *t*
$\vec{P}_i$	Coefficent of correlation matrix
SAA	Sample Average Approximation
t	Index for time period
TCCI	Total Constant Commonality Index
$\vec{\mu}$	Vector consisting of the means of the multivariate normal distribution
$\vec{V}$	Vector consisting of the demands of components from period t to period t+L-1
$\vec{W}$	Vector consisting of the demands of products from period t+L to period t+L+l
X_t	System state before the allocation decision at period *t* (the nominal path)
$\mathrm{X}_t^{'}$	System state before the allocation decision at period *t* for the perturbed path
Y_j	Order-up-to level of component *j*
$Y_{j,t}$	Order-up-to level of component *j* at period *t* (the nominal path)
$Y_{j,t}^{'}$	Order-up-to level of component *j* at period *t* for the perturbed path

$\vec{Y}$	Vector of the components' order-up-to levels
$\vec{Y}_t$	Vector of the components' order-up-to levels at the beginning of period *t* (the nominal path)
$\vec{Y}_t^{'}$	Vector of the components' order-up-to levels at the beginning of period *t* for the perturbed path
$\Delta Y_{j,t-L_j}$	Magnitude of the perturbation on $Y_{j,t-L_j}$
$Z_{i,t}$	Standardized inventory position at the end of period *t*
$z_{i,t}$	Deviation of the demand at period *t*
α_i	Probability of running out of component stock for product *i*
$\varepsilon_{i,t}$	White noise from the auto-regressive process of product *i* at period *t*
$\lambda_{j,t}$	Lagrange multipliers of the component *j* constraint
$\lambda_{a_i,t}$	Lagrange multipliers of the non-negative allocation constraint of product *i*
$\gamma_{i,k}$	Autocovariance of product *i* at lag *k*
$\phi_{i,k}$	Weight parameters of the past value $z_{i,t-k}$ in autoregressive process of product *i*
$\vec{\phi}_i$	Column vector of $\phi_{i,j}$ with AR(P)
$\rho_{i,k}$	Autocorrelation of product *i* at lag *k*
$\vec{\rho}_i$	Column vector of $\rho_{i,k}$
$\psi_{i,k}$	Weighted parameters of the value of the white noise $\varepsilon_{i,t-k}$
ω_n	The n-th sample path

Ω	All possible sample paths
σ_i	Standard deviation of demand for product i
σ'^2_i	Homogeneous variances of the white noise for product i
$\sum_D$	Variance-covariance matrix of the multivariate normal distribution
Σ_U	Normalized variance-covariance matrix for the components
$\theta_{j,t}$	Quantity of component j that is unassigned at the end of period t
$\theta_{i,t}$	Quantity of product i that is unassigned at the end of period t, which represents the number of component sets that can used to assemble product i
$\vec{\theta}$	Vector consisting of the quantities of the unassigned components on-hand
ν_i	Safety factor of product i after the allocation
ν	Equal safety factor

CHAPTER 1 INTRODUCTION

1.0 Background

Assemble-To-Stock (ATS) or Make-To-Stock (MTS) manufacturing is the norm for a very wide range of industries, including electronic gadgets, sporting goods, tools, toys, home computing accessories (Najarian, 2006) retail products, cars, appliances, silicon chips, machine tools (McCutcheon 1994); Bertsimas and Paschalidis, 1999), critical repair operations, such as aircraft components (Reeve and Srinivasan, 2005) and Vendor-Managed Inventory hub (Lee, 2004). An ATS system is generally practiced when customers turn to other substitute products in the event of unavailability of their requested items; when the assembly lead time is longer than the customer's delivery requirement; when there are fewer product options with relatively long life cycles; when there is expected seasonal hike in demands with limited assembly capacity; or when there is a contractual agreement with a customer that certain stock levels must be maintained and high penalty costs for any breach of the agreed service levels are stipulated. Due to the contract agreement, the contract manufacturers in the upstream of a typical supply chain are pressurized to keep huge inventory in order to provide sufficient flexibility to the downstream which constitutes to the imbalance in inventory level (Lee, 2006). Accordingly, an inventory of product is required to buffer demand fluctuation. For example, a hard disk drive manufacturer adopts an ATS system because his customers, mainly personal computer (PC) makers, demand the availability of the hard disk drives for shipments at the stage of orders placement. A hard disk drive is typically assembled from four main components: head stack assembly, disks platter, printed circuit board (PCB) and spindle motor. The manufacturer sources these components globally. It generally takes one or more weeks for the delivery of these components from overseas suppliers, and the assembly, testing and packaging processes take another day or two. For the same reasons, ATS systems are commonly employed in chip fabrication / PCB assembly (Grotzinger et al., 1993) and other components / subassemblies (Mohebbi and Choobineh, 2005),

Due to the economic benefits gained through better inventory management (Orlicky, 1979), reduced new product development cost, scale economies in material or component costs, (Fisher et al., 1999; Thonemann and Brandeau, 2000; Swaminathan, 2001) and improvement in forecast accuracy due to demand aggregation (Dogramaci, 1979), common components, where the components are common to a set of distinct products, are widely used in a typical assembly process. Using common components allows the manufacturer to pool his inventories and minimize his risk exposure when demand is variable. For instance, firms have reduced their risk in the procurement of the common components in PC systems by aggregating the demand based on the requirements from all the products and then purchasing the required components (Swaminathan, J. M., 2001). In other words, quantities of common components can be pooled together and then allocated to the respective products at the assembly stage to minimize total costs or alleviate shortage problems. We refer to this as component-sharing.

In ATS systems, demand is uncertain at the time of placing order quantities with suppliers, as well as during the allocation decision to determine the quantities of products to be made, and correspondingly the quantities of components to be released into assembly lines. Exploiting commonality in reducing total costs is a strategically important goal in procurement and allocation policies to explicitly address the complexity arising from demand uncertainty. A procurement policy determines the order quantity of components, while a component allocation policy decides the quantities of components to be allocated to each product and released into the assembly line to manufacture that product. Failure to take into account component commonality and demand uncertainty results in higher inventory costs by having excessive inventory on-hand for some products and insufficient for others.

1.1 Scope and Purpose of the Study

This research work intends to provide greater insight into the behavior of the system by analyzing several procurement and allocation policies that take advantage of component-sharing under a two-echelon ATS system. Each product is assembled from several common components. A periodic review policy is assumed. The process of procuring components and their subsequent allocation to products, where both processes have lead times associated with them, arises in the system. Ordered components, arriving after a certain lead time, may be stored for future use or released for assembly into various products in accordance with the allocation decision, with the end-products then available to meet demand after the completion of the assembly process. The demand for product is stochastic but its probability distribution is known. Procurement and allocation decisions that exploit component-sharing are devised to reduce the total cost.

This research work can be divided into three main phases. The first two phases focus on allocation policy and the last on procurement policy. We first apply probabilistic analysis to evaluate the performance of a particular component allocation policy, known as equal fractile allocation policy, by comparing it with another allocation policy that does not allow component-sharing. The optimal constant order-up-to level of component is determined for both policies. The comparison between both allocation policies is based on the optimum safety factor which indirectly reflects the level of safety stock required. The equal fractile allocation policy allocates the components in such a way that, after the allocation, all products have equal probability of running out of stock. Hence, the policy requires the system to withhold the release of components to some products, even if all the necessary components are available, because backlogs exist for a different product. This assumption reduces the practical applicability but leads to algebraic simplifications. We examine conditions when the equal fractile allocation policy may be chosen over the other policy, and vice versa. In this probabilistic analysis, other assumptions have been included to make the model mathematically tractable. For instance, the demand for different products is uncorrelated and the allocation policies try to 'push' or release immediately all components into assembly lines in all periods.

We then examine another allocation policy which considers total cost and can be easily implemented into a real manufacturing system. This allocation policy aims to minimize the total cost given the system state information, and is subject to component availability and other constraints. We call it a myopic allocation policy because it only looks at the total cost, of holding excess components or products and paying a penalty on backlogged demands, for a fixed period while making the allocation decision. For component procurement, a constant order-up-to level is used where orders are placed to raise the inventory position of each component back to its respective level at every review period. Due to the complexity in the dependency of every period, a closed-form solution to determine the optimum order-up-to levels in total cost minimization is unavailable. With such system complexity being generally outside the scope of analytical models, we adopt a simulation-based optimization approach to help in the search for optimum order-up-to levels. Using Infinitesimal Perturbation Analysis (IPA), we estimate the gradient or derivative of total costs with respect to order-up-to levels of components. We then use the information on the gradient estimation to adjust the order-up-to level accordingly via a steepest descent algorithm. The performance of the myopic allocation policy is compared with two allocation policies that do not allow the sharing of common components, namely the pure push allocation policy and the two-echelon allocation policy. Under the pure push allocation policy, the manufacturer only holds end-product inventory. Components, on arrival at the manufacturer's premises, are 'pushed' to assembly lines. There is no on-hand component, unassigned to any product, at the end of every period, and therefore there is no unassigned component to be stored. The two-echelon allocation policy is similar to the pure push policy, except that there is a possibility of unassigned on-hand components at the end of every period, based on total cost minimization, and therefore these unassigned components are kept in the store for future allocation. By comparing the pure push policy with the two-echelon policy, we can measure the benefit of the echelon effect, allowing the manufacturer to keep components that are unassigned and not released to the assembly line; while by comparing the myopic allocation policy against the two-echelon policy, we can quantify the benefit of component-sharing.

Finally, we draw our attention to different procurement policies under correlated demands. The product demand follows an auto-regressive process, but is independent of other product demands. Forecast demands are updated every review period based on the latest information. We analyze the performance of having constant order-up-to level over dynamic order-up-to level. These two policies employ a myopic allocation policy to minimize the total cost of a period when the component allocation decisions are made. The allocation rule synchronizes the allocation decision to exploit component-sharing based on the availability of the components required. In addition to the effect of dynamic order-up-to level over constant order-up-to level, we evaluate the benefits of component-sharing by measuring the dynamic order-up-to level with sharing against the dynamic order-up-to level without sharing. We solve the problem of determining the dynamic order-up-to level by minimizing the average total cost which is estimated by a Monte Carlo simulation-based technique called Sample Average Approximation (SAA). Through several numerical examples, we illustrate the benefits of component-sharing and the effect of dynamic level over the constant level. Figure 1.1 summarizes the research approach in addressing the component commonality in ATS systems.

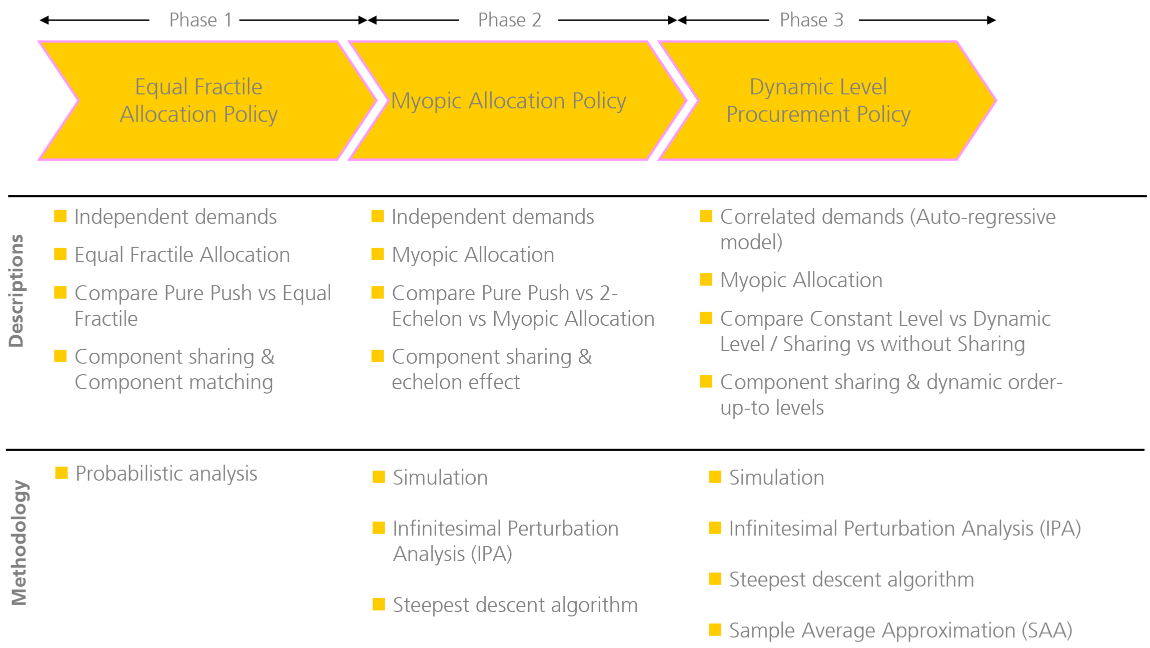


Figure 1.1: Research methodology by phases

In summary, the main contributions of this research work include the ability to:

- Explore and quantify the benefits of component-sharing in an ATS system
- Propose a component allocation policy which exploits component commonality and minimizes the total cost in an ATS system
- Propose a dynamic procurement policy that minimizes the total cost when the demands for finished goods are auto-correlated over time.
- Combine a few simulation-based optimization techniques to identify the optimum allocation quantities for the components allocation policy and the optimum procurement quantities for the dynamic procurement policy.

1.2 Organization of the Dissertation

The organization of this thesis is as follows. Chapter 2 reviews previous related literature on the component commonality, Assemble-To-Order (ATO), Assemble-to-

Stock (ATS), Component Allocation Policy. Generally, prior research effort in the analysis of the benefits of component commonality in an ATS environment is limited. Chapter 3 describes the two-echelon system, and the modeling and analysis of the equal fractile allocation policy which allocates components based on safety factors. This chapter also studies the effect of risk pooling and component matching. Chapter 4 formulates and analyzes the myopic allocation policy which minimizes the total cost. It also describes the IPA method used for the gradient estimation and steepest descent algorithm to seek the optimum order-up-to levels when the myopic allocation policy is utilized. Chapter 5 focuses on evaluating the effect of component-sharing under correlated demands by comparing the constant order-up-to level and the dynamic order-up-to level. As the demands are correlated over time, the impact on the dynamic order-up-to levels can be analyzed based on the latest updates of the forecasted demands. Chapter 6 provides a summary of the findings.

CHAPTER 2: LITERATURE REVIEW

2.0 Introduction

An overview of the publications on component commonality is provided in this chapter. The area of work varies, including: the impact of commonality on the workload of a manufacturing system (Collier, 1981; Guerrero, 1985; Vakharia et al., 1996); the effect of component commonality on the component costs through activity-based costing framework (Labro 2004); the trade-off between the decreased logistics costs and loss of risk-pooling benefits in plant networks which spread component manufacturing over each plant as compared to those that consolidate component manufacturing in a single plant (Kulkarni et. al 2005): the effect of commonality in a multi-level production-inventory system with a fixed planning horizon and deterministic demand following the Lot-for-Lot ordering principle and the Fixed Order Quantity policy (Zhou and Grubbstrom, 2004); the benefits of introducing commonality at a production stage or delaying the point of differentiation as much as possible (Lee et al., 1993; Lee, 1996; Lee and Tang, 1999; Carg and Tang ,1997; Whang and Lee, 1999; Ma et al., 2002); the optimal configuration of common components in a modularity approach, called a vanilla box in this literature, and their inventory levels / costs (Swaminathan and Tayur, 1998); the impact of modularity on product reliability and maintainability aspects (Biegel and Bulcha, 1976); deploying a nine-step methodology for developing Component commonality for desensitizing the product application and meting the functionality objectives of the product. (Majerus, et. al 2000); the application of commonality to a system on the design efficiency. Commonality can increase the component weight, volume, and power consumption which become progressively more severe as commonality is increased. (Thomas, 1992); the change in product configuration due to component commonality on the supply chain design (Ghosh et al. 2005); the effect of setup costs, component commonality, and capacity on schedule stability in a lot-sizing based production planning model (Meixell, 2005) and the product design problems of incorporating commonality (Rutenberg, 1969; Starr ,1965; O'Reilly, 1975; Fisher et al., 1999). In the context of this research, the focus is the effect of component commonality on the procurement policies and the allocation policies to reduce the safety stock level or the total cost.

2.1 General Review

The effect of component commonality in an Assemble-To-Order (ATO) environment has been described extensively in the literature. In the earlier work, Collier (1981, 1982) defines an index to measure the degree of component commonality. He shows the impact of the degree of commonality index on the aggregate safety stocks, inventory cost, set-up cost, and work load variability through a single-period model. Collier (1982) provides proofs on the benefits of component commonality, that the standard deviation of aggregated demands is less than the sum of standard deviations when the demands being aggregated are independent, which is risk-pooling. Lin et al. (2000) apply the model in their research. McClain et al. (1984) make some comments on the assumptions in Collier's model (1982) and Collier (1984) gives further explanations to clarify ambiguities. Moscato (1972, 1976) proposes the use of entropy to measure the degree of commonality. Moscato (1972) and Shaftel (1977) provide an extensive list of benefits involved in modularization, a way to achieve component commonality. Wacker and Trevelan (1986) propose a newly defined commonality index, where 0 represents no commonality and 1 represents complete commonality.

A general one-echelon, single-period model with uniform demand distribution is formulated by Baker (1985). He examines the impact of component commonality and correlated demands on the safety stocks based on safety factors. He points out that an equal safety factor approach cannot be applied when determining the required stock level for common components to obtain the same aggregate service standard, the probability of meeting all demands simultaneously, due to the unavailability of certain components to make the product. The aggregate service standard becomes multidimensional in the presence of component commonality. Baker et al. (1986) address a similar inventory-minimizing model using two products, each needing two unique components. While maintaining the same aggregate service standard, they study the effect on the total stock of replacing the one unique component in both products with a common component. The results reveal that:

a. The total inventory required to meet a specified service standard decreases with the degree of commonality,
b. The stock level of one common component is lower than the aggregate sum of the unique components it replaces, and
c. The stock level of unique components increases with the degree of commonality because the effective overstocking cost is reduced due to risk-pooling.

Gerchak et al. (1988) use a similar model, but allow an arbitrary number of products and general joint demand distribution. They highlight that the finding (a) is always true, while (b) and (c) only hold if the costs of unique components are equal and all products share the same common component. It may not necessarily hold for the scenario of unequal holding costs of unique components. Eynan (1996) extends Baker et al.'s model by studying the impact on the costs of demand correlation between different products and the saving that might be realized with commonality. Uniform demands are used as they result in closed-form solutions that can be easily analyzed. Following intuition, the smaller the correlation, the larger the saving as the risk-pooling is more effective. These models, developed by Baker (1985), Baker et al. (1986), Gerchak et al. (1988) and Eynan (1996), are restricted to one common component shared by many products.

Bagchi and Gutierrez (1992) maximize the service level for exponential and geometric demand distribution, and found that the aggregated stock requirement decreases at an increasing rate with commonality and the marginal cost reduction increases with commonality. Eynan and Rosenblatt (1996) look into the effects of increasing the degree of commonality. They demonstrate that component commonality may not always bring lower inventory cost when the common component is more expensive than the unique components it replaces. The common component may cost more because it must be at least as reliable as the components it replaces, and may even carry extra redundant features resulting from standardization. They provide conditions for when to use common components, taking account of the increase in component cost. Hillier (1999b) and Cheung (2000) extend it to a multi-period model.

Jonsson and Silver (1989) consider a single-period model with any number of products that are assembled from a number of components. The objective is to determine the allocation of a given budget among the components to minimize the expected unit shortage when the demand for products follows normal independent distribution. They show that there is no closed-form solution for the problem, and develop heuristics and bounding procedures which produce excellent near-optimum results. The objective function is proved to be convex in Fu and Fong (1998) for any continuous demand distribution in a one common component and two unique components' scenario, where there are two products each requiring one unique component and sharing one common component.

The following researchers use a profit maximization model to quantify the increase in profit by introducing common components into ATO systems. Sauer (1984) explores the effect of commonality on the stock levels of the unique components using a single period, with two products sharing a common component as well as requiring unique components. Gerchak and Henig (1986) generalize the work by Sauer (1984) to an arbitrary number of products and components. The authors prove that profit increases when more commonality is introduced, and component commonality causes the optimal stock levels of the unique components to rise. For optimality, the single-period model can be extended to an infinite horizon model, with independent and identically distributed demands, by modifying the model parameters (Heyman and Sobel 1984). Hillier (2000) utilizes Gerchak and Henig's result (1989) to develop heuristics to determine good stocking levels and limits on cost for a multi-period model with any number of products and general demand distributions. Rudi (2000) provides an analytical view of the optimal component stock levels for a single-period profit maximization problem in a two-product case. Tayur (1995) considered a two-level, periodic review, finite horizon ATO problem with multiple products that have random demands and share several common components, but capturing the delivery lead times in his model. The focus of this research is on simulation-based stochastic approximation with a number of sample paths to minimize the total costs of finite T-period. Srinivasan et al. (1998) formulate a multi-period cost minimization model with several products sharing several common components subject to service standard constraints. However, only approximations of the model are solved heuristically, using standard non-linear programming methods. Agrawal and Cohen (2001), Gallien and Wein (2003) and Lu et al (2003) consider stochastic procurement lead times for components.

Hillier (1999a) considers the replacement of all components with one common component to reduce the inventory cost in a multiple-period model with general demand distributions. Eynan and Rosenblatt (1997) and Hillier (2002) propose using both cheaper unique components and a more expensive common component. Initially, all demand is met with unique components; when stock of any unique component runs out, the demand is met by the common component. Eynan and Rosenblatt (1997) consider only a single-period model with two products and uniform distribution, and service standard constraints. Hillier (2002) develops a multiple-period model with purchasing and inventory costs. Hillier (1999b, 2000) has shown that a more expensive common component is often not worthwhile in the multi-period scenario.

Betts and Johnson (2005) consider JIT replenishment and component substitution policy decisions under the finite capital investment. A simplified ATO system with multi-product and single level bill of material is analyzed. They show that the substitution reduces the inventory safety stock due to risk pooling but can increase the capital cost tied up if a higher cost component is used for substitution.

Mohebbi and Choobineh (2005) employ simulation techniques to investigate the impact of increasing component commonality in ATO systems when there is uncertainty in product demands and component procurement lead times.

Akcay and Xu (2004) formulate a two-stage stochastic integer program to determine the optimal base-stock policy and the optimal component allocation policy for the ATO system. They show that the component allocation problem is a general multidimensional knapsack problem and is NP-hard. They propose a simple, order-based component allocation rule and show that the model can be solved in either polynomial or pseudo polynomial time.

Earlier publications on component commonality in ATO systems cannot be applied directly in our research model due to the following considerations:

- Demand Uncertainty. In ATO systems, components are allocated to fulfill actual demands. In ATS systems, components are allocated based on forecasted demands.
- Component Allocation. In ATO systems, when the components available exceed the actual demand, the allocation of the components will be equivalent to the actual demand. On the other hand, when the components are insufficient to meet the demand, we adopt a component rationing policy to identify the product in which the components should be allocated in priority such as giving priority to product of smaller demand (Baker et al., 1986; Gerchak et al., 1988). In ATS systems, components are allocated based on forecasted demands to

optimize the pre-defined objective functions which may be cost or service level related.

- Dependency of Allocation Decision. In ATO systems, actual demands are realized before the allocation of components is determined. Thus, the allocation decisions are independent over periods. In ATS systems, the quantities that are previously allocated and are still in-process are taken into account. Thus, the dependency of allocation decisions becomes more complicated if demands are time-correlated.

Very few publications directly address component commonality under an ATS system. Guerrero (1985) studies the effect of component commonality on the production and inventory costs by comparing three production planning strategies. They are Make-To-Stock, Make-To-Order and Hybrid Assemble-To-Order, which is a combination of Make-To-Stock and Make-To-Order. Various combinations of Wagner-Whitin (WW) and lot-for-lot sizing are used. This research is on the production scheduling problem which can be formulated as a Mixed Integer Program (MIP). The demands are known from the master production schedule when solving the MIP problem.

Baker et al. (1986) and Gerchak (1988) do consider ATS systems. In a single-period problem, without considering delivery lead times and assembly lead times, an ATO system holds stocks at the component level, where the demands for common components can be aggregated or pooled together, while an ATS system holds inventory of finished products only, where all products are assembled before the demand is realized. Due to the nature of the problem considered, the ATS system can be represented by a system with all unique components. There is no benefit from component-sharing in ATS systems in this setting.

Grotzinger et al. (1993) consider a single component shared by *N* different products in a two-echelon multi-period system. Given an allocation policy for assigning the components to the products, the paper computes the optimal order-up-to level of component which minimizes the inventories, but still satisfies the aggregate service standard constraint. The authors simplify the analysis by considering the delivery and production lead time to be of one period.

Most publications assume that the procurement lead time is zero (or negligible). With longer lead times, the benefits of component commonality would increase due to the higher safety stock required to cater for the longer protection interval. The closest publication to our work is Grotzinger et al. (1993). However, we consider several components shared by several products in a two-echelon Assemble-To-Stock system. In addition in our research, the delivery lead time and the assembly lead time are multiple integers of the review period; the demands for products are stochastic and a periodic review policy is assumed; and we analyze procurement and allocation policies that incorporate component-sharing and compare them with policies without component-sharing.

2.2 Component Allocation Policy

The majority of the research in this field is directed towards obtaining an optimal component allocation policy for the system, so that either

- A desired standard of service is achieved with the minimum inventory investment, or

- The service standard is maximized under a given inventory budget, or
- The total cost of inventory holding and penalty cost is minimized.

With component commonality, the component allocation decisions should take into account the availability of the other components required to complete a product. The current literature on the allocation decisions of a component depend on the product demand for that component alone and are independent of the allocation decisions made for other components. Zhang (1997) studies a system similar to that of Hausman et al. (1998). He is interested in determining the order-up-to level of each component that minimizes the inventory cost, subject to a service standard requirement for each product. Zhang proposes the fixed priority component allocation rule, under which all the product demands requiring a given component are assigned a predetermined priority order, and the available inventory of the component is allocated accordingly. Product rewards and marketing policies are among factors used to establish the priority order. Agrawal and Cohen (2000) investigate the fair shares scheme to allocate the available stock of components to product orders, independent of the availability of the other required components. The quantity of the component allocated to a product is determined by the ratio of the realized demand of that product to the total realized demand of all the product orders. They derive the optimal component stock levels that minimize the total inventory cost, subject to product service standard requirements. The policies proposed by Zhang (1997) and Agrawal and Cohen (2000) fail to take into consideration the availability of other components needed to complete the assembly when solving the model or making the component allocation decision.

CHAPTER 3 EQUAL FRACTILE ALLOCATION POLICY

3.0 Introduction

This chapter investigates the impact of component-sharing in a two-echelon Assemble-To-Stock system consisting of several common components and end-products. While there are benefits of risk-pooling when component-sharing is allowed, a component-matching problem may occur due to demand uncertainty. We study these conflicting effects by comparing an equal fractile allocation policy with a policy that does not allow component-sharing. The equal fractile allocation policy allocates the components in such a way that, after the allocation, all products have equal probability of running out of stock. We also look at a special scenario of the equal fractile allocation policy with inventory cost considerations and show that this problem can be reduced to a newsvendor problem. In Section 3.1, we examine the risk-pooling effect and the component-matching effect by comparing the equal fractile allocation policy with the pure push policy. Section 3.2 looks at the conditions when the benefit of risk-pooling prevails over component-matching. In Section 3.3, the inventory cost model of the equal fractile allocation policy is considered. Section 3.4 provides a summary.

3.1 System Description and Model Development

Consider a manufacturer producing I number of products. These products are assembled from a combination of J number of components, which are ordered from a supplier, with identical lead time for all the components.

When the components arrive, they are released into the assembly process. All components required for the assembly of a product must be available before the assembly process starts. L denotes the delivery lead time of component, and l denotes the assembly lead time. A schematic diagram of the model is shown in Figure 3.1.

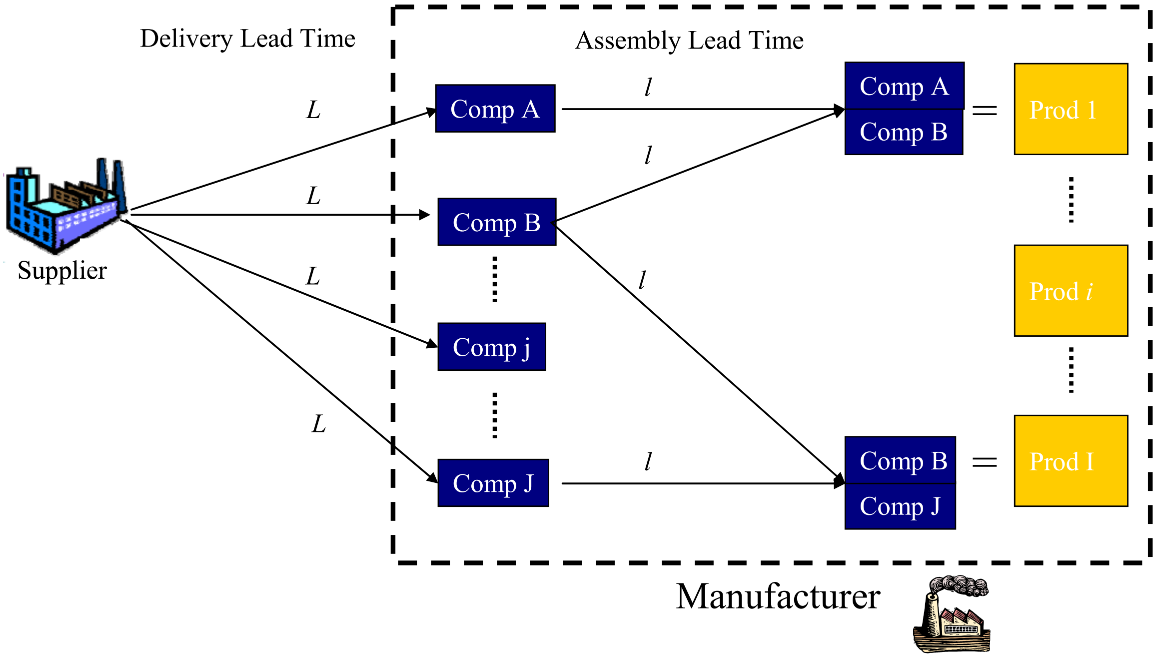


Figure 3.1: A two-echelon supply chain system

We assume the following:

a. The suppliers' and the manufacturer's production capacities are unlimited.

b. The unfilled demands are back-ordered.

c. The demands for products in every period are random, independent and follow a multivariate normal distribution with means that given a vector $\vec{\mu}$ and a variance-covariance matrix $\sum_D$. The demands are not correlated over time.

d. The quantities of components received by the manufacturer are sufficient to be allocated to each product so that equal probability of running out of stock can be achieved for all products (Eppen and Schrage, 1981).

e. The delivery lead times of the components from the suppliers to the manufacturer are deterministic and identical.

f. The assembly lead time is deterministic and identical for all the products.

g. *L* and *l* are multiple integers of the review period.

Please take note that in reality the model may have different delivery lead time for different components and different assembly lead time for different products. The above will be further discussed in our proposed policy in Chapter 4 However, for the ease of tractability and comparison between the policies, we have assumed that all components and products enjoy the same delivery lead time and assembly lead time respectively.

3.1.1 Pure Push Policy

If we do not allow sharing of common components, the order-up-to levels of products, denoted by a vector $\vec{S}$, with equal safety factor *K* for all products, are

$$\vec{S} = (L + l + 1)\vec{\mu} + \sqrt{(L + l + 1)}K\sum_{i=1}^{I}\sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}\,\vec{e}_i \qquad (3.1)$$

where

$\vec{e}_i$ is the *i*th standard basis vector of vector space $\mathbf{R}^I$. (The standard basis for $\mathbf{R}^I$. consists of *I* elements. $\vec{e}_1 = (1,0,0,...0)$; $\vec{e}_2 = (0,1,0,...0)$,… $\vec{e}_I = (0,0,0,...,1)$ in which each $\vec{e}_i$ has 1 for its *i*th component and 0 for every other component (Jennings, 2004)) .

$\sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}$ is the standard deviation of demand for product *i*, σ_i .

The optimal $\vec{S}$ has elements which are constants.

To enhance a fairer comparison with the equal fractile allocation policy that will be introduced in the next sub-section, we have assumed the marginal probability (instead of the joint probability) of the safety factor, *K*, to be the same for all products.

3.1.2 Equal Fractile Allocation Policy

In a two-echelon supply chain system where component-sharing is allowed, the quantities of the common components, originally ordered and pre-allocated to make the prescribed quantities of different products, can be pooled together, and these quantities can be allocated to products when they arrive at the manufacturer's premises to cope with demand fluctuation. The allocation decision also creates a component-matching problem, i.e. if a component required to assemble a particular product is not available. It may not always be possible to release or 'push' immediately all common components to the assembly process. Component-sharing may result in some 'excess' components which cannot be allocated due to the component-matching effect. In this scenario, risk-pooling the demands of common components may not always be beneficial since the benefit could be offset by the component-matching effect. To our knowledge, there is no literature that simultaneously reviews these conflicting effects. However, Gerchak and Mossman (1992), Kim (2002) and Benjaafar and Kim (2001) have shown that risk-pooling may not necessarily be beneficial in other types of inventory problems. Gerchak and Mossman use a single-period model to show that when several random demands of a product are aggregated, the order quantity of the aggregated demands is increased to achieve the same targeted service level if it falls below a certain limit. For instance, for the same service standard to be attained, the optimal order quantity from the aggregated inventory model, of two demand distributions, is higher than the sum of two optimal order quantities from the individual inventory model. In addition, when service standard is low, risk-pooling effect moves the order quantities away from mean demand or median demand. Kim

(2002) and Benjaafar and Kim (2001) consider a production inventory system with several types of items where each item is managed according to its own base-stock policy. The results show that the benefits of risk-pooling can be significant when utilization rate of the product capacity is moderate. However, the results will become insignificant when the utilization rate is on the extreme high or low end.

We show how we derive the model of the equal fractile allocation policy of common components to determine the service standard of product *i* given order-up-to levels. To manage the inventory, we track the inventory position in units of component. The inventory position of the component includes the pipeline inventory from the supplier to the manufacturer, the inventory kept at the component level, the inventory in the assembly process, and also the net inventory of the products. Vector $\vec{Y}$ denotes the components' order-up-to levels to protect against demand uncertainty taking account of the delivery lead time, the assembly lead time and the review period. Since an ATS system is considered, we assume that the manufacturer tries to release all components into the assembly process.

Let matrix $\vec{G}$ denote the product structure, where G_{ij} is 1 when product *i* uses component *j*, assuming only one unit of component *j* is used, and 0 otherwise. Denote $\vec{V}$ as a vector consisting of the demands of components from period *t* to period $t+L-1$ and $\vec{W}$ as a vector consisting of the demands of products from period $t+L$ to period $t+L+l$, which are expressed as

$$\vec{V} = \sum_{k=t}^{t+L-1} \vec{G}^T \vec{D}_k \qquad (3.2)$$

and

$$\vec{W} = \sum_{k=t+L}^{t+L+l} \vec{D}_k \tag{3.3}$$

respectively, where $\vec{V} \sim N\left(L\vec{G}^T\vec{\mu}, L\vec{G}^T\Sigma_D\vec{G}\right)$ and $\vec{W} \sim N((l+1)\vec{\mu},(l+1)\Sigma_D)$.

At the beginning of period t, the manufacturer places an order to increase the inventory positions of components to $\vec{Y}$. The order arrives at the beginning of period $t+L$. The quantities of components are allocated to the respective products so that the inventory positions of products, i.e. the sum of the inventory in the assembly process and the net inventory of products, can protect against the products' demand uncertainty for $l+1$ periods, which is from period $t+L$ to period $t+L+l$. After the allocation, the inventory positions of products are

$$\vec{s} = (l+1)\vec{\mu} + \sqrt{(l+1)}\sum_{i=1}^{I} \nu_i \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}\,\vec{e}_i \tag{3.4}$$

where ν_i is the safety factor of product i.

If the equal fractile allocation policy is used, the quantities of components are allocated such that all products have equal probability of running out of stock at the end of period $t+L+l$. Let denote that $\nu_i = \nu \ \forall i$. At the beginning of period $t+L$, the quantities of components available at the manufacturer's site, including the finished products, are $\vec{Y} - \vec{V}$, which is obtained through subtracting those components in transit from supplier to manufacturer from the order-up-to levels, where $\vec{V}$ is given in (3.2). These quantities can be obtained by summing up those components that are in the assembly lines, those that have become part of finished products, and those components that are unassigned to any product. These unassigned components cannot be assigned and released into the assembly process due to the component-matching problem. Denote $\vec{\theta}$ as a vector consisting of the quantities of the unassigned components on hand. Then we have the following relation

$$\vec{Y} - \vec{V} = \vec{G}^T \vec{s} + \vec{\theta} \tag{3.5}$$

Since it is impossible for the quantities of the unassigned components to be negative, the quantities must be at least greater than zero, $\vec{\theta} \geq \vec{0}$ (zero vector). Equation (3.5) can be simplified to inequalities

$$Y_j - V_j \geq \left(\vec{G}\vec{e}_j\right)^T \vec{s} \qquad \forall j \tag{3.6}$$

where Y_j is the jth element of the vector $\vec{Y}$, V_j is the jth element of the vector $\vec{V}$, $V_j = \sum_{k=t+1}^{t+L} \left(\vec{G}\vec{e}_j\right)^T \vec{D}_k \sim N\left(L\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}, L\left(\vec{G}\vec{e}_j\right)^T \Sigma_D \left(\vec{G}\vec{e}_j\right)\right)$.

By replacing the $\vec{s}$ with (3.4) and substituting $v_i = v \ \ \forall i$ in (3.6), we have

$$Y_j - V_j \geq \left(\vec{G}\vec{e}_j\right)^T \vec{s} = (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + \sqrt{(l+1)}v\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}\vec{e}_i \qquad \forall j \tag{3.7}$$

or

$$v \leq \frac{Y_j - V_j - (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}}{\sqrt{(l+1)}\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}\vec{e}_i} \qquad \forall j \tag{3.8}$$

The equal safety factor selected must be able to satisfy equation (3.8) for all components.

Hence, the maximum value of ν is the smallest possible value of ν that satisfies all the constraints given in equation (3.8),

$$v = \min\left\{ \frac{Y_j - V_j - (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}}{\sqrt{(l+1)}\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}\vec{e}_i}, j = 1,\ldots,J \right\} \tag{3.9}$$

From Equation (3.4), the inventory positions of product i after the equal fractile allocation is

$$s_i = (l+1)\mu_i + \sqrt{(l+1)}\nu\sigma_i$$

where σ_i is the standard deviation of the demand of product i.

By subtracting from it the product demands of product i from period $t+L$ through $t+L+1$, the net inventory of product i at the end of period $t+L+l$ is

$$X_i = (l+1)\mu_i + \sqrt{(l+1)}\nu\sigma_i - W_i$$

By substituting (3.9) into (3.4) into the above equation, we have

$$X_i = (l+1)\mu_i + \min\left\{\frac{Y_j - V_j - (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}}{\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i}, j = 1,\ldots,J\right\}\sigma_i - W_i \tag{3.10}$$

The net inventory of product i is a random variable as the demand for products W_i and the demands for components V_j are uncertain.

Given the distribution function of X_i and the values of Y_j, j = 1,2,…,J, the service standard of product i can be determined as follows

$$1\text{-}\alpha_i = \Pr(X_i \geq 0 \mid \vec{Y}) = 1 - F_{X_i}(0 \mid \vec{Y}) \tag{3.11}$$

where α_i is the probability of running out of stock for product i. $F_{X_i}(\bullet \mid \vec{Y})$ is the conditional cumulative distribution function of X_i.

3.2 Comparison between the Two Policies

We compare the two policies by looking at the difference in the component order-up-to levels or equivalent safety stock levels to achieve the same service standard. The procedure used for the comparison is as follows: we consider a given order-up-to level with equal safety factor for the pure push policy. This is equivalent to setting a particular safety factor in (3.1), say $K = K_1$, if equal service standards for all products are assumed. The order-up-to level of component j can be expressed as

$$Y_j^{(1)} = \left(\vec{G}\vec{e}_j\right)^T \vec{S} = (L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + K_1\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T \left(\sum_{i=1}^{I}\sqrt{\vec{e}_i^T\Sigma_D\vec{e}_i}\vec{e}_i\right) \forall j \quad (3.12)$$

From this given $Y_j^{(1)}$, we determine the corresponding service level, 1-α, if the equal fractile allocation policy is applied. By substituting (3.12) into (3.9), we have

$$\nu = \min\left\{\frac{(L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + K_1\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T \left(\sum_{i=1}^{I}\sqrt{\vec{e}_i^T\Sigma_D\vec{e}_i}\vec{e}_i\right) - V_j - (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}}{\sqrt{(l+1)}\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I}\sqrt{\vec{e}_i^T\Sigma_D\vec{e}_i}\vec{e}_i},\right.$$
$$\left. j = 1,2,3,,......J\right\}$$

$$= \min\left\{\frac{K_1\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T \left(\sum_{i=1}^{I}\sqrt{\vec{e}_i^T\Sigma_D\vec{e}_i}\vec{e}_i\right) + L\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} - V_j}{\sqrt{(l+1)}\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I}\sqrt{\vec{e}_i^T\Sigma_D\vec{e}_i}\vec{e}_i}, j = 1,\ldots,J\right\}$$

Let $V_j = L\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + \xi_j$

where $\xi_j \sim N[0, L\left(\vec{G}\vec{e}_j\right)^T \Sigma_D \left(\vec{G}\vec{e}_j\right)]$.

Since $V_j = \sum_{k=t+1}^{t+L}\left(\vec{G}\vec{e}_j\right)^T \vec{D}_k$ which is normally distributed with mean $L\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}$ and variance $\left(L\left(\vec{G}\vec{e}_j\right)^T \Sigma_D \left(\vec{G}\vec{e}_j\right)\right)$

Substituting V_j into the equation,

$$\nu = \min\left\{ \frac{K_1\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T\left(\sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i\right)+L\left(\vec{G}\vec{e}_j\right)^T\vec{\mu}-L\left(\vec{G}\vec{e}_j\right)^T\vec{\mu}-\xi_j}{\sqrt{(l+1)}\left(\vec{G}\vec{e}_j\right)^T\sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i}, j=1,\ldots,J \right\}$$

$$= \min\left\{ \frac{K_1\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T\left(\sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i\right)-\xi_j}{\sqrt{(l+1)}\left(\vec{G}\vec{e}_j\right)^T\sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i}, j=1,\ldots,J \right\}$$

$$= K_1\sqrt{\frac{L+l+1}{l+1}} - \max\left\{ \frac{\xi_j}{\sqrt{l+1}\left(\vec{G}\vec{e}_j\right)^T\sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i}, j=1,\ldots,J \right\} \tag{3.13}$$

Note that (3.13) is independent of $\vec{\mu}$.

Let

$$\eta = \max\{u_j; j=1,\ldots,J\} \tag{3.14}$$

where $u_j = \dfrac{\xi_j}{\left(\vec{G}\vec{e}_j\right)^T\sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i}$.

If the demands between products are independent and there is component-sharing, u_j s are positively correlated. By substituting (3.12) into (3.10), it can be reduced to

$$X_i = (l+1)\mu_i + \left(K_1\sqrt{L+l+1} - \max\left\{ \frac{\xi_j}{\left(\vec{G}\vec{e}_j\right)^T\sum_{i=1}^{I}\sqrt{\vec{e}_i^{\,T}\Sigma_D\vec{e}_i}\,\vec{e}_i}, j=1,\ldots,J \right\} \right)\sigma_i - W_i$$

By dividing it by σ_i and from the equation given in (3.14),

$$X_i/\sigma_i = (l+1)\mu_i/\sigma_i + K_1\sqrt{L+l+1} - \eta - W_i/\sigma_i$$

Let, $W_i/\sigma_i = (l+1)\mu_i/\sigma_i + \vartheta$

Since $W_j = \sum_{k=t+L}^{t+L+l} (\vec{e}_i)^T \vec{D}_k$ which is normally distributed with mean (l+1) μ_i and variance $(l+1)\sigma_i^2$

where $\vartheta \sim N(0, l+1)$.

The equation can be simplified as

$$X_i / \sigma_i = K_1 \sqrt{L+l+1} - \eta - \vartheta \tag{3.15}$$

Hence, the service standard of products for the equal fractile allocation policy is

$$
\begin{aligned}
1-\alpha &= \Pr(X_i \geq 0 \mid K_1) \\
&= \Pr(X_i / \sigma_i \geq 0 \mid K_1) \\
&= \Pr\left(K_1 \sqrt{L+l+1} - \eta - \vartheta \geq 0 \mid K_1\right) \\
&= \Pr\left(\eta + \vartheta \leq K_1 \sqrt{L+l+1}\right) \qquad \text{(from (3.15))} \qquad (3.16) \\
&= \iint_{n+z \leq K_1 \sqrt{L+l+1}} f_\eta(n) f_\vartheta(z) dndz \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{K_1 \sqrt{L+l+1} - z} f_\eta(n) f_\vartheta(z) dndz \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{K_1 \sqrt{L+l+1} - z} f_\eta(n) dn f_\vartheta(z) dz \\
&= \int_{-\infty}^{+\infty} F_\eta\left(K_1 \sqrt{L+l+1} - z\right) f_\vartheta(z) dz \qquad (3.17)
\end{aligned}
$$

Since the random variables η and ϑ are independent of each other

where $F_\eta(x) = \int_{-\infty}^{x} \ldots \int_{-\infty}^{x} f(u_j; j = 1,\ldots,J) du_1 \ldots du_J$, $f_\eta(x) = f(u_j; j = 1,\ldots,J)$ and $f_\vartheta(z)$ is the probability density function for ϑ . Note that $f(u_j;\ j = 1,\ldots,J)$ is the joint probability density function of random variables u_j , j = 1,…,J given in (3.14) and this function follows a multivariate normal distribution with means of zero and variance-covariance matrix $\Sigma_u = L(\vec{G}')^T \Sigma_D \vec{G}'$ where $\vec{G}' = \sum_j \left(\frac{\vec{G}\vec{e}_j}{\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i \vec{e}_i}} \right) \vec{e}_j^T$.

Based on this service standard, we find the equivalent component order-up-to levels for the pure push policy, which are represented by

$$Y_j^{(2)} = \left(\vec{G}\vec{e}_j\right)^T \vec{S} = (L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + K_2\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T \left(\sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i \vec{e}_i}\right) \forall j \tag{3.18}$$

where $K_2 = \Phi^{-1}(1-\alpha)$ (3.19)

and $\Phi^{-1}(\bullet)$ is the inverse of standard normal cumulative function.

The effectiveness of the equal fractile allocation policy is measured by the difference in the component order-up-to levels of the two policies. Hence, by subtracting (3.12) from (3.18), we have

$$Y_j^{(2)} - Y_j^{(1)} = \left(K_2 - K_1\right)\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T \left(\sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i \vec{e}_i}\right) \qquad \forall j$$

and the difference can be normalized as

$$\frac{Y_j^{(2)} - Y_j^{(1)}}{\sqrt{(L+l+1)}\left(\vec{G}\vec{e}_j\right)^T \left(\sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i \vec{e}_i}\right)} = \left(K_2 - K_1\right) \qquad \forall j \tag{3.20}$$

Let $\Delta = K_2 - K_1$ (3.21)

Clearly, if $\Delta > 0$, the equal fractile allocation policy fares better since the pure push policy requires more safety stock to attain the same service standard.

Next, we show that under certain conditions, the equal fractile allocation policy always performs better than the pure push policy, which is $\Delta \geq 0$. In order to show these conditions, we first develop a lower bound for the service standard of products for the equal fractile allocation policy, $\Pr\left(\eta + \vartheta \leq K_1\sqrt{L+l+1}\right)$ which is given in Lemma 3.1.

Lemma 3.1.

When u_j $\forall j$ are positively correlated, $\left(\Phi(\beta K_1)\right)^J$ is the lower bound of $\Pr\left(\eta + \vartheta \leq K_1\sqrt{L+l+1}\right)$

$$\text{where } \beta = \min \left\{ \sqrt{\frac{L+l+1}{\left(\frac{\left(\hat{G}\hat{e}_j\right)^T \Sigma_D \left(\hat{G}\hat{e}_j\right)}{\left(\hat{G}\hat{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\hat{e}_i^T \Sigma_D \hat{e}_i}\,\hat{e}_i}\right) L+l+1}},\ j = 1,2,3\ldots.J \right\}. \tag{3.22}$$

Proof

Replacing η in (3.16) with (3.14), the service standard of products for the equal fractile allocation policy is written as

$$\Pr\left(\eta + \vartheta \leq K_1\sqrt{L+l+1}\right)$$

$$= \Pr\left(\max\{u_j; j = 1,\ldots,J\} + \vartheta \leq K_1\sqrt{L+l+1}\right)$$

$$= \Pr\left(\max\{u_j + \vartheta \leq K_1\sqrt{L+l+1}; j = 1,\ldots,J\}\right)$$

$$= \Pr\left(u_1 + \vartheta \leq K_1\sqrt{L+l+1}, u_j + \vartheta \leq K_1\sqrt{L+l+1}, \ldots u_J + \vartheta \leq K_1\sqrt{L+l+1}\right) \tag{3.23}$$

Let E_j be the event when $u_j + \vartheta \leq K_1\sqrt{L+l+1}$. $\Pr(E_j)$ be the probability of event E_j occurs, (3.23) can be written as

$= \Pr(E_1 E_2 E_3 ....E_J)$

$= \Pr(E_1)\Pr(E_2 \mid E_1)\Pr(E_3 \mid E_1 E_2).........\Pr(E_J \mid E_1 E_2 ...E_{J-1})$

(Multiplication rule, Ross 1976, pp. 71)

As $u_j \quad \forall j$ are positively correlated,

$\Pr(E_2 \mid E_1) \geq \Pr(E_2)$ (The probability of E_2 happens given the occurrence of E_1 is higher as both events are positively correlated) and

$\Pr(E_j \mid E_1 E_2 ...E_{j-1}) \geq \Pr(E_j) \qquad \forall j \geq 2$

Hence,

$\Pr(E_1 E_2 E_3 ....E_J) \geq \Pr(E_1)\Pr(E_2)....\Pr(E_J)$

Hence, we can infer that

$\Pr\left(\max\{u_j; j = 1,..., J\} + \vartheta \leq K_1 \sqrt{L + l + 1}\right)$

$\geq \prod_j \Pr\left(u_j + \vartheta \leq K_1 \sqrt{L + l + 1}\right)$

$$= \prod_j \Phi\left(K_1 \sqrt{\frac{L + l + 1}{\left(\frac{\left(\hat{G}\hat{e}_j\right)^T \Sigma_D \left(\hat{G}\hat{e}_j\right)}{\left(\hat{G}\hat{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\hat{e}_i^T \Sigma_D \hat{e}_i}\, \hat{e}_i} \right) L + l + 1}} \right)$$

$> \left(\Phi(\beta K_1)\right)^J$

As $0 \leq \Phi(\bullet) \leq 1$ □

(This bound can also be inferred from Lehmann, 1966, when $u_j \quad \forall j$ are positively correlated)

Theorem 3.1

Given any β as in (3.22) > 1, $J \geq 2$ and u_j are positively correlated, there exists a value of $K_1^*(\beta, J)$ such that Δ as given in (3.21) = $K_2 - K_1 \geq 0$ when $K_1 \geq K_1^*(\beta, J)$.

<u>Proof</u>

To show that $\Delta = K_2 - K_1 \geq 0$, it is sufficient to show that

$$\Delta' = \Pr\left(\eta + \vartheta \leq K_1\sqrt{L+l+1}\right) - \Phi(K_1) \geq 0 \qquad (3.24)$$

where $\Pr\left(\eta + \vartheta \leq K_1\sqrt{L+l+1}\right)$ represents the service standard of the equal fractile policy and $\Phi(K_1)$ represents the service standard of the pure push policy.

Equation (3.21) implies that the better service standard is achieved by the equal fractile allocation policy given both policies have an equal safety factor.

Let $\overline{\Phi}(\cdot) = (1 - \Phi(\cdot))$. From Lemma 3.1, $\Pr\left(\eta + \vartheta \leq K_1\sqrt{L+l+1}\right) \geq (\Phi(\beta K_1))^J$

$$\Delta' \geq \left(1 - \overline{\Phi}(\beta K_1)\right)^J - \left(1 - \overline{\Phi}(K_1)\right)$$

Using $\left(1 - \overline{\Phi}(\beta K_1)\right)^J = \sum_{m=0}^{J} \binom{J}{m} \left(-\overline{\Phi}(\beta K_1)\right)^{J-m}$ (Ross, 1998)

$$\Delta' \geq 1 - J\left(\overline{\Phi}(\beta K_1)\right) + O\left(\overline{\Phi}(\beta K_1)\right)^2 - \left(1 - \overline{\Phi}(K_1)\right)$$

where $O\left(\overline{\Phi}(\beta K_1)\right)^2$ contains second and higher orders of $\overline{\Phi}(\beta K_1)$.

$$\Delta' \geq \overline{\Phi}(K_1)\left(1 - \frac{J\overline{\Phi}(\beta K_1)}{\overline{\Phi}(K_1)} + \frac{O\left(\overline{\Phi}(\beta K_1)\right)^2}{\overline{\Phi}(K_1)}\right) \qquad (3.25)$$

$$\frac{\overline{\Phi}(\beta K_1)}{\overline{\Phi}(K_1)} = \frac{\int_{\beta K_1}^{\infty} e^{-x^2/2} dx}{\int_{K_1}^{\infty} e^{-x^2/2} dx} \le \frac{\int_{\beta K_1}^{\infty} e^{-x} dx}{\int_{K_1}^{\infty} e^{-x} dx} \qquad K_1 \ge 1$$

because $e^{-(x^2/2)}$ decreases at a faster rate than e^{-x} as x increases after the point of inflexion which is equal to 1.

When $K_1 \ge 1$, in order to have $\Delta' \ge 0$,

$$\left(1 - \frac{J\overline{\Phi}(\beta K_1)}{\overline{\Phi}(K_1)} + \frac{O\left(\overline{\Phi}(\beta K_1)\right)^2}{\overline{\Phi}(K_1)}\right) \ge 0 \qquad \text{as } \overline{\Phi}(K_1) \ge 0$$

or

$$\left(1 - \frac{J\overline{\Phi}(\beta K_1)}{\overline{\Phi}(K_1)}\right) \ge 0$$

as $O\left(\overline{\Phi}(\beta K_1)\right)^2$ is small and positive when high service standard is required.

$$\frac{\overline{\Phi}(\beta K_1)}{\overline{\Phi}(K_1)} \le \frac{1}{\mathrm{J}}$$

Hence

$$\frac{\overline{\Phi}(\beta K_1)}{\overline{\Phi}(K_1)} = \frac{\int_{\beta K_1}^{\infty} e^{-x^2/2} dx}{\int_{K_1}^{\infty} e^{-x^2/2} dx} \le \frac{\int_{\beta K_1}^{\infty} e^{-x} dx}{\int_{K_1}^{\infty} e^{-x} dx} \le \frac{1}{\mathrm{J}}$$

$$\frac{-\left[e^{-x}\right]_{\beta K_1}^{\infty}}{-\left[e^{-x}\right]_{K_1}^{\infty}} \le \frac{1}{\mathrm{J}}$$

$$e^{-K_1(\beta-1)} \le \frac{1}{\mathrm{J}}$$

$$-K_1(\beta-1) \le -\ln J$$

$$K_1 \geq \frac{\ln J}{(\beta - 1)}$$

So, $\Delta^{'} \geq 0$ and thus $\Delta \geq 0$

when $K_1 \geq K_1^*(\beta, J) = \max\left\{\frac{\ln J}{(\beta - 1)}, 1\right\}$ (3.26)

This theorem gives a lower bound of which the equal fractile allocation policy will be always favorable when the safety factor K_1 exceeds this limit. □

β indicates the benefit that we can get from the 'least-shared' component type if the equal fractile allocation policy is applied. $\beta > 1$ means that each type of component must be shared by at least two products to allow risk-pooling of the demands for components. If $\beta = 1$, it means that the product structure has at least one type of component that is unique (not shared by different products). In this case, the equal fractile allocation policy is not favorable which can be shown in Lemma 3.2.

Lemma 3.2.

If β (3.22) = 1, then Δ (3.21) ≤ 0.

<u>Proof</u>

Let component 1 be unique and so, $u_1 \sim N(0,L)$

From (3.16), the service standard of equal fractile allocation policy

$$\Pr\left(\max\{u_j; j = 1,...,J\} + \vartheta \leq K_1\sqrt{L + l + 1}\right)$$

$$\leq \Pr\left(u_1 + \vartheta \leq K_1\sqrt{L + l + 1}\right)$$

(which can be easily inferred from $\Pr(E_1 E_2 E_3 ....E_J) \leq \Pr(E_1)$)

$$= \Phi(K_1)$$

Hence, $\Pr\left(\eta + \vartheta \le K_1 \sqrt{L + l + 1}\right)$ - $\Phi(K_1) \le 0$ and thereby $\Delta \ \le 0$

□

3.3 Numerical Analysis

In this section, we construct several scenarios to study the effect of the equal fractile allocation policy. First, we use a simple scenario to illustrate the risk-pooling effect and the component-matching effect. We then look at a two-component and three-product scenario and subsequently extend it to a higher number of components and products. In the analysis, Σ_D is a diagonal matrix as the demands are independent.

3.3.1 Effect of Risk-Pooling and Component-Matching

Consider an example of one common component shared by two products. For the pure push policy, the system can be viewed as two individual independent subsystems as illustrated in Figure 3.2(a). With the equal fractile allocation policy, the quantities of component are shared to fulfill the demands of the two products as shown in Figure 3.2(b). The equal fractile allocation policy developed in this scenario can be reduced to the model developed by Eppen and Schrage (1981), which has the following relation

$$K_2 \qquad = K_1 \frac{\sqrt{L+l+1}(\sigma_1 + \sigma_2)}{\sqrt{L(\sigma_1^2 + \sigma_1^2) + (l+1)(\sigma_1 + \sigma_2)^2}} \qquad (3.27)$$

Equation (3.24) shows that $K_2 \ge K_1$ when $K_1 \ge 0$ which illustrates the benefit of risk-pooling.

To illustrate the effect of component-matching, we introduce an additional component (component 2) for the assembly of product 2, which is shown in Figure 3.2(c) and 3.2(d). Figure 3.2(c) represents the pure push policy while Figure 3.2(d) represents the equal fractile allocation policy. Component-matching is introduced because one unit of component 2 is required to 'match' with one unit of component 1 to produce one unit of product 2. The perfect matching of components may not always be possible for the equal fractile allocation policy since it requires every product to have equal probability of running out of stock.

From the product structure given in Figure 3.2(d), (3.14) can be reduced to

$$\eta = \max\{u_1, u_2\} \tag{3.28}$$

where $u_1 = \xi_1 / (\sigma_1 + \sigma_2)$ and $u_2 = \xi_2 / \sigma_2$.

Note that the joint probability density function of u_1 and u_2, which is denoted by $f(u_1, u_2)$, follows a bivariate normal distribution with mean (0,0)T and variance-covariance matrix $\begin{bmatrix} L(\sigma_1^2 + \sigma_2^2)/(\sigma_1 + \sigma_2)^2 & L\sigma_2^2/(\sigma_1 + \sigma_2)(\sigma_2) \\ L\sigma_2^2/(\sigma_1 + \sigma_2)(\sigma_2) & L \end{bmatrix}$. The cdf of η is given by

$$F_\eta(z) = \int_{-\infty}^{z} \int_{-\infty}^{z} f(u_1, u_2) du_1 du_2 \tag{3.29}$$

Consequently, the service standard and K_2 can be obtained from (3.17), (3.19) and (3.29).

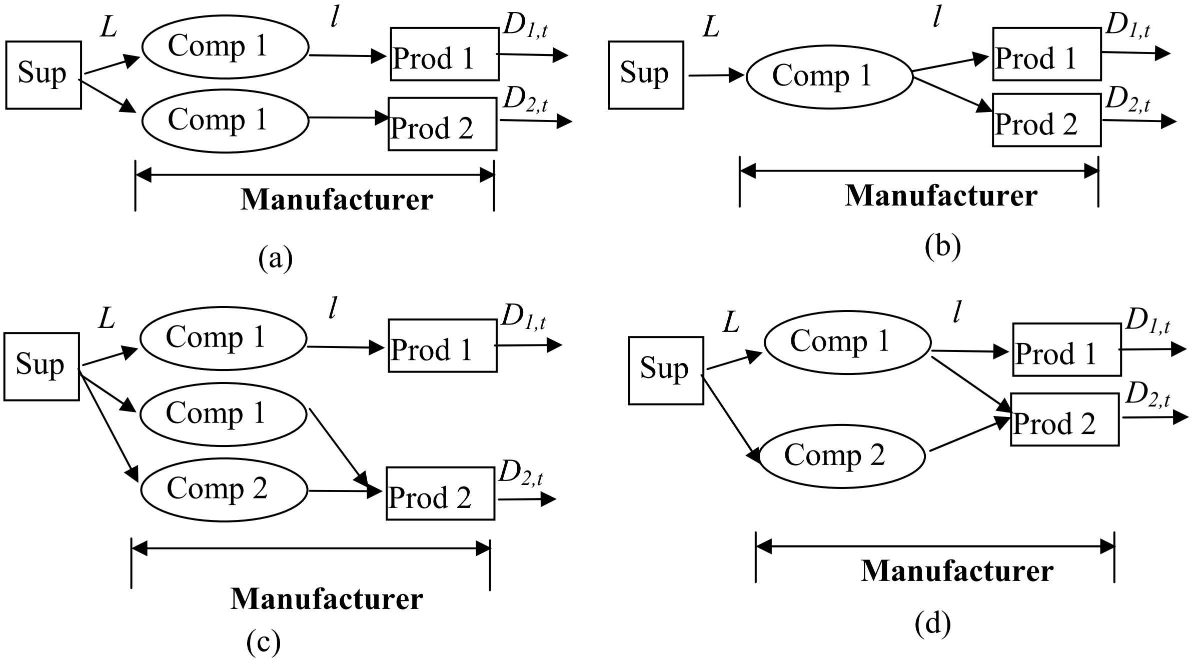


Figure 3.2: One / two components, two products scenario

By setting $\sigma_1 = \sigma_2 = 1$, $L = 5$ and $l = 1$, the numerical analysis is performed to evaluate the effect of risk-pooling (systems in Figure 3.2(a) and 3.2(b)) and component-matching (systems in Figure 3.2(c) and 3.2(d)). Figure 3.3 presents the trend of Δ with respect to K_1. It shows that the benefit of risk-pooling increases with K_1. From the graph of component-matching, the values of Δ are negative due to the negative impact of component-matching. However, the values of Δ improve as K_1 increases. The equal fractile allocation policy is always inferior to the pure push policy, because there is a unique component 2 in the product structure, $\beta = 1$. In this scenario, the equal fractile allocation policy is always unfavorable, which has been explained and proved in Lemma 3.2.

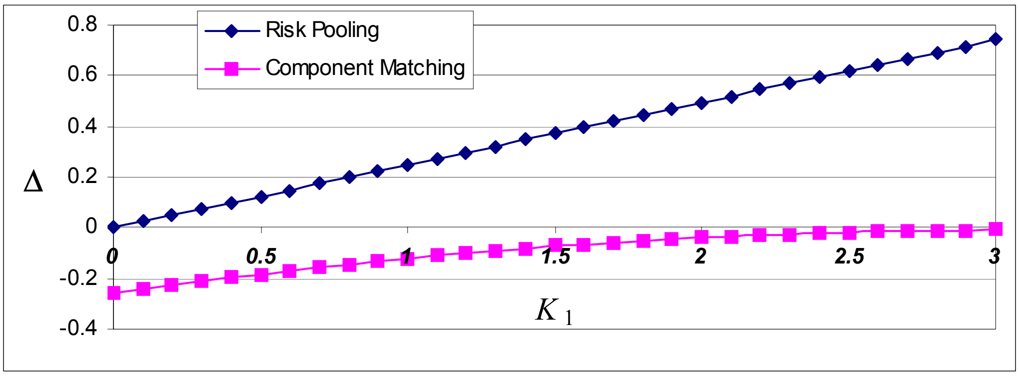


Figure 3.3: Δ versus K_1 for risk-pooling and component-matching

3.3.2 Analysis of Two Components and Three Products Scenario

We further extend our analysis to a two common components and three products scenario as shown in Figure 3.4. The parameters are $u_1 = \xi_1/(\sigma_1+\sigma_2)$ and $u_2 = \xi_2/(\sigma_2+\sigma_3)$ while the variance-covariance matrix of the joint probability density function, $f(u_1,u_2)$, is $\begin{bmatrix} L(\sigma_1^2+\sigma_2^2)/(\sigma_1+\sigma_2)^2 & L\sigma_2^2/(\sigma_1+\sigma_2)(\sigma_2+\sigma_3) \\ L\sigma_2^2/(\sigma_1+\sigma_2)(\sigma_2+\sigma_3) & L(\sigma_2^2+\sigma_3^2)/(\sigma_2+\sigma_3)^2 \end{bmatrix}$.

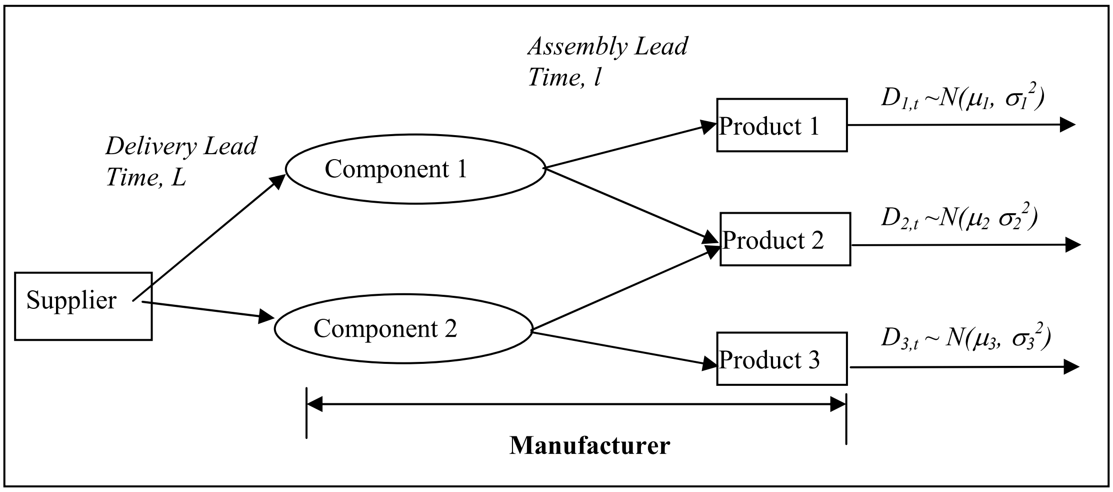


Figure 3.4: A two-component three-product scenario

We evaluate the equal fractile allocation policy with $\sigma_1 = \sigma_3 = 1$ and $l = 1$ while varying L and σ_2. Figure 3.5 shows that the equal fractile allocation policy tends to perform better when K_1 is high. The higher K_1, the better is the equal fractile allocation policy. When K_1 is high, the gain through the risk-pooling of common components can be used to compensate the loss due to the component mismatch. When K_1 is low, the loss in the component-matching effect outweighs the gain in the risk-pooling effect. When the ratio of L/l increases, the effects of component-matching and risk-pooling are magnified. The sign of Δ depends on the value of K_1, which has been proven in Theorem 3.1.

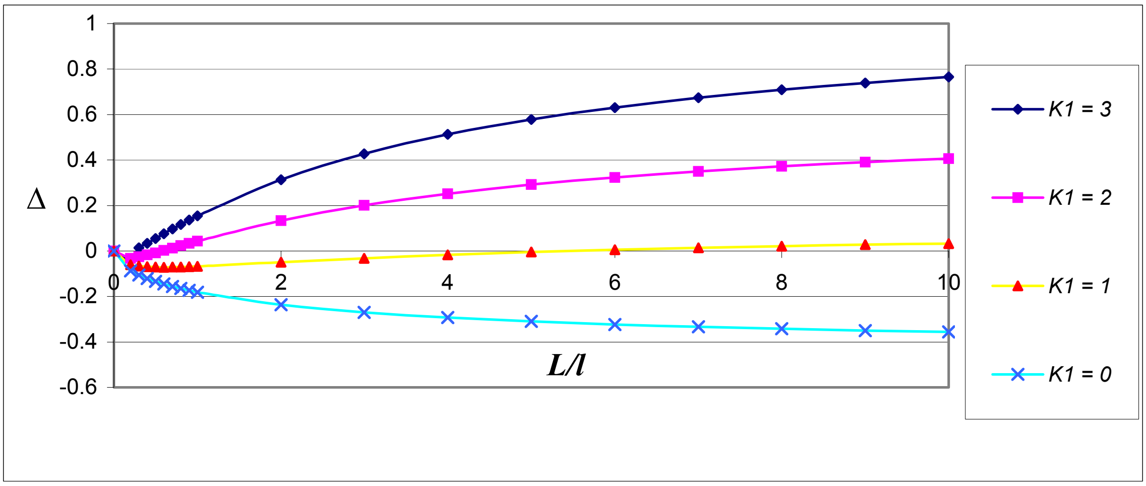


Figure 3.5: Δ versus L/l ratio for different K_1 when $\sigma_2/\sigma_1 = 1$

Figure 3.6 illustrates the effect of σ_2/σ_1 on Δ under different combinations of L/l and K_1 values. For the given values of L/l and K_1, the maximum value of Δ occurs where σ_2/σ_1 is around the value of 1. As σ_2/σ_1 deviates away from 1, the saving derived from the component-sharing diminishes. This is because, when the ratio of σ_2/σ_1 either increases or decreases, the component-matching problem is more likely to occur while the effect of risk-pooling decreases. Therefore, more safety stock is required to maintain the same service standard. In the extreme scenario, where the ratio of σ_2/σ_1 is very large, the equal fractile allocation policy converges to the pure push policy since the two-echelon system is dominated by the demands of product 2, and therefore the effects of risk-pooling and component-matching become insignificant. In another extreme scenario, where the ratio of σ_2/σ_1 is close to 0, the values of Δ are negative. When the demand of product 2 has little or no variability, it lessens the possibility of component-sharing and thus the risk-pooling effect is minimized. The result is consistent with Lemma 3.2.

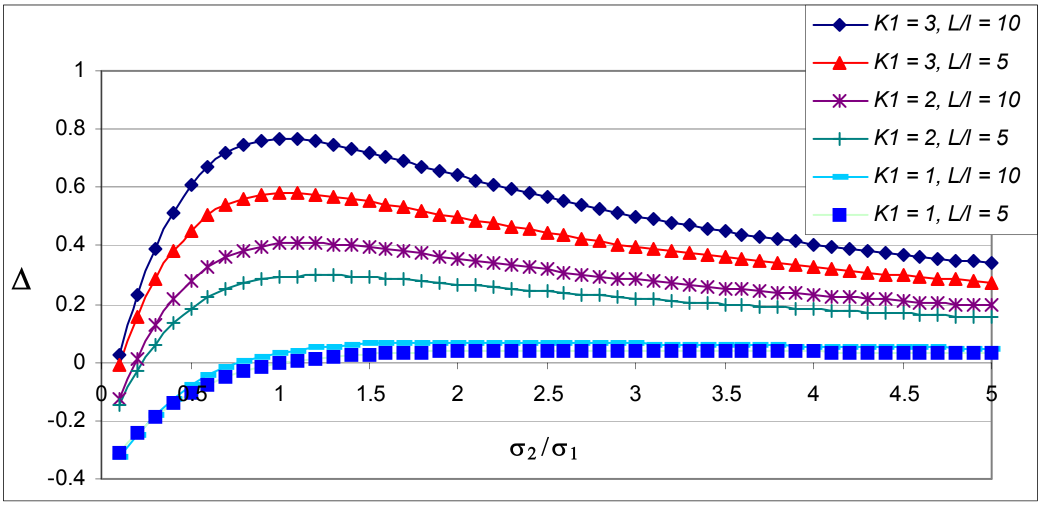


Figure 3.6: Δ versus σ_2/σ_1

Note that if we approximate the distribution of η to follow a normal distribution, we could derive an explicit relationship between K_1 and K_2. Appendix A gives the details of the derivation and the mathematical expression. In fact, the solution obtained from the approximation method is very close to the solution derived from numerical analysis.

3.3.3 Analysis of Higher Number of Components and Products

We extend the previous analysis to a higher dimension of components and products. For illustration, we look at an assembly system of five common components and five products to provide a greater insight into the impact of the degree of commonality. There are 225 possible combinations of product structure. In this research, we restrict our attention to a special scenario where the number of components required to assemble a product is the same for all products and the number of products that use a component is the same for all components. We use the work done by Wacker and Treleven (1986) to define the degree of commonality.

$$\text{TCCI} = 1 - \frac{J-1}{\sum_{j=1}^{J} A_j - 1} \tag{3.30}$$

where TCCI is defined as Total Constant Commonality Index, J is the total number of components and A_j is the number of immediate successors of component j (number of products that needs component j). The limit of TCCI ranges from 0 to 1, where 0 represents no common component being used and 1 represents complete commonality. For example, for five components and five products, and $A_j = 2$ $\forall j$, then the TCCI is

$$\text{TCCI} = 1 - \frac{5-1}{2(5)-1} = 5/9$$

Define P_i as the number of immediate predecessors of product i (number of components needed to assemble one unit of product i). Note that the solutions for different combinations of product structure are equal if $P_i = A_j = n$ $\forall i, j$ (by symmetry) and the product structure cannot be decomposed into a subset of two or more independent product structures.

By setting $\sigma_i = \sigma \quad \forall i$, the results are consistent with the earlier finding as shown in Figure 3.8. Given a TCCI value, the equal fractile allocation policy performs better as K_1 increases which has more safety stock to provide avenues for component-sharing. The graphs also indicate that as TCCI increases, the value of Δ increases for a given K_1 value. This trend is expected since a higher degree of commonality provides more opportunities for risk-pooling. In addition, increasing *L/l* could amplify the effects of risk-pooling and component-matching.

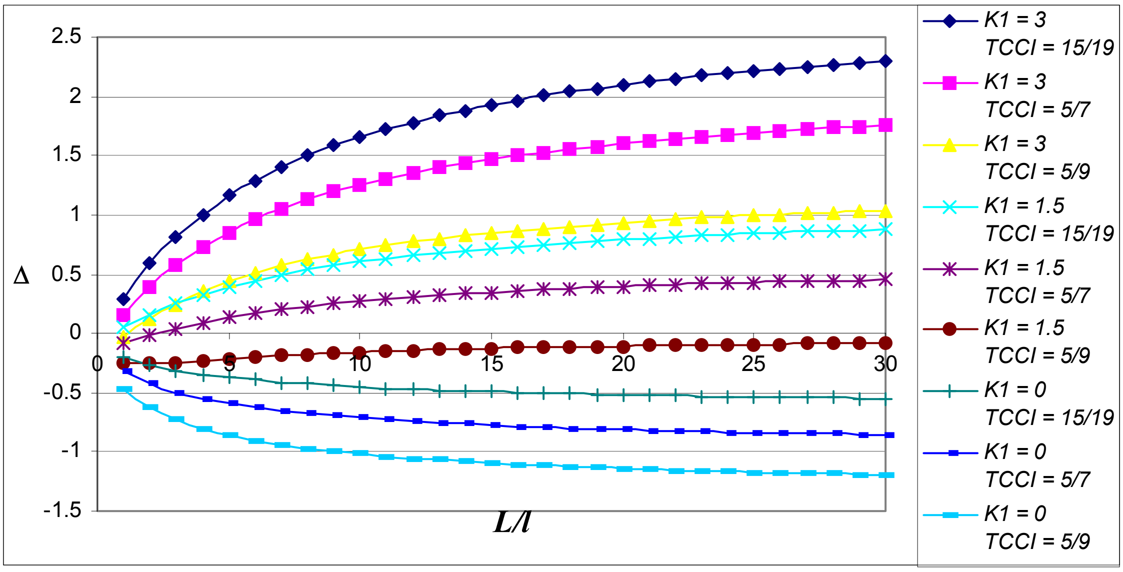


Figure 3.7: Δ versus *L/l* ratio for different TCCI and K_1

The graph of Δ versus K_1 for a given β is shown in Figure 3.8. When $L = 5$, β equals to 1.247, 1.382 and 1.468 respectively for TCCI values of 5/9, 5/7 and 15/19. A larger β correlates to a higher TCCI. The results show that the benefit of risk-pooling increases when either β increases (TCCI increases) or K_1 increases.

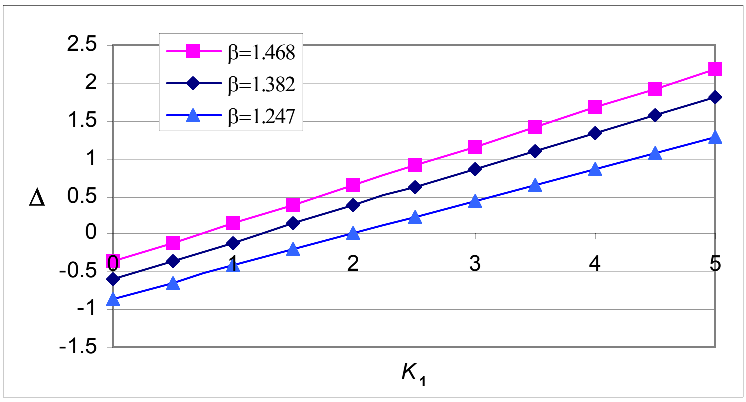


Figure 3.8: Δ versus K_1 for different β

As the total number of components increases, the numerical analysis becomes tedious and possibly inaccurate because of the convolution of many distribution functions as given in (3.17). Clarke (1961) has provided a method to approximate the distribution function of η with sufficient accuracy. This method provides an efficient way to estimate the safety factor, K_2, when the system consists of a large number of components.

3.4 Cost Model

In this section, we develop a total cost model and show that the optimal order-up-to levels, $\vec{Y}^*$, based on the equal fractile allocation policy can be easily computed under certain conditions. The total cost involves the holding cost of component j, h_{c_j}, and the holding cost of product i, h_i, that are charged at the end of every period for every unit of inventory remaining and a penalty cost, p_i, that is charged at the end of every period for every unit short of product i. The expected total cost per period is

$$\sum_i \left(h'_i\ I^+_{i,t+L+l} + p_i I^-_{i,t+L+l}\right) + \sum_j \left(h_{c_j} \left(Y_j - V_j + \sum_i G_{ij} I^-_{i,t+L+l} \right)\right) \text{ or}$$

$$\sum_i \left(h_i^{'} \int_0^{\infty} x_i f_{X_i}(x_i) dx_i + \left(p_i + \sum_j G_{ij} h_{c_j} \right) \int_{-\infty}^{0} (-x_i) f_{X_i}(x_i) dx_i \right) + E\left[\sum_j h_{c_j} \left(Y_j - V_j \right) \right] \quad (3.31)$$

where $h_i^{'} = h_i - \sum_j G_{ij} h_{c_j} \quad \forall i$ and $h_i \geq \sum_j G_{ij} h_{c_j} \quad \forall i$; G_{ij} is the quantity of component j used to assemble one unit of product i; $I^{+}_{i,t+L+l}$ ($I^{-}_{i,t+L+l}$) is the amount of inventory on-hand (backlogged demand) for product i at the end of period $t+L+l$; Y_j is the order-up-to level of component j; V_j is the quantity of component j in transit from supplier, $V_j \sim N\left(L\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}, L\left(\vec{G}\vec{e}_j\right)^T \Sigma_D \left(\vec{G}\vec{e}_j\right) \right)$; $f_{X_i}(x_i)$ is the probability density function of inventory on-hand for product i at the end of period $t+L+l$. There is no holding cost for the components in transit from the supplier.

The optimal order-up-to-levels that minimize the cost given in (3.31) are not easily found. There is no closed-form solution, and computing the optimal levels would require numerical local search techniques or approximation methods.

However, in order to provide some insights on the optimality of the order-up-to levels, we consider a special scenario:

a. Equal inventory holding cost and equal penalty cost ($h_i^{'} = h^{'}$, $p_i = p \ \forall i$ & $h_{c_j} = h_c \ \forall j$)

b. The number of different components required to assemble a product is the same for all end-products

c. The variance of the demand per period is the same for all end-products. ($\sigma_i^2 = \sigma^2 \ \forall$i)

d. The sum of the row for the variance-covariance matrix, Σ_U is the same for all components.

One example of (d) is the symmetric product structure.

Let m denote the number of different components required to assemble a product. With the equal fractile allocation policy, the net inventory of product i as given in (3.10) can be simplified as follows

$$X_i = (l+1)\mu_i + \min\left\{ \frac{Y_j - V_j - (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}}{\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}\,\vec{e}_i}, j = 1,\ldots,J \right\}\sigma_i - W_i$$

Replacing $\left(\vec{G}\vec{e}_j\right)^T \sum_{i=1}^{I} \sqrt{\vec{e}_i^T \Sigma_D \vec{e}_i}\,\vec{e}_i = m\sigma_i$

$$X_i = (l+1)\mu_i + \min\left\{ \frac{Y_j - V_j - (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}}{m}, j = 1,\ldots,J \right\} - W_i \qquad (3.32)$$

Let $Q_j = Y_j$ - $(L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}$. As $(L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}$ is the expected demands which cover the delivery lead time, assembly lead time and order review period, the quantity Q_j represents the amount of safety stock for component j.

Substituting $V_j = L\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + \xi_j$ and $W_i = (l+1)\,\mu_i + \vartheta'$ into (3.32),

$$X_i = (l+1)\mu_i + \min\left\{ \frac{Y_j - (L)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} - \xi_j - (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}}{m}, j = 1,\ldots,J \right\} - (l+1)\mu_1 + \vartheta'$$

$$X_i = \min\left\{ \frac{Q_j - \xi_j}{m}, j = 1,\ldots,J \right\} - \vartheta' \qquad (3.33)$$

where $\vartheta' \sim N(0, (l+1)\sigma^2)$; $\xi_j \sim N[0, L\left(\vec{G}\vec{e}_j\right)^T \Sigma_D \left(\vec{G}\vec{e}_j\right)]$.

Let

$$\eta^{'} = \max\left\{\frac{\xi_j - Q_j}{m}, j = 1,...,J\right\} \tag{3.34}$$

and $X' = \eta^{'} + \vartheta^{'}$. Then $X_i = -X'$. Let $\vec{Q}$ be the vector of Q_j. It can be easily observed that X_i $\forall i$ follow the same distribution and $\eta^{'}$ follows a multivariate normal distribution with means, $-\vec{Q}/m$, and variance-covariance matrix, Σ_U (where $u_j = \xi_j / m$). This matrix can be viewed as the normalized variance-covariance matrix for the components). Note that the difference between $\eta^{'}$ and η is that the means of $\eta^{'}$ are equal to $-\vec{Q}/m$, while the means of η are equal to $\vec{0}$. The distribution of X_i is given by

$$\begin{aligned}
F_{X_i}(x) &= \Pr(X_i \le x) \\
&= \Pr(-X' \le x) \\
&= \Pr(X' \ge -x) \\
&= 1 - \Pr(X' \le -x) = 1 - F_{X'}(-x) \\
&= 1 - \Pr(\eta' + \vartheta' \le -x) \\
&= 1 - \iint_{n+z \le -x} f_{\eta'}(n) f_{\vartheta'}(z) dndz \\
&= 1 - \int_{-\infty}^{+\infty} \int_{-\infty}^{-x-z} f_{\eta'}(n) f_{\vartheta'}(z) dndz \\
&= 1 - \int_{-\infty}^{+\infty} \int_{-\infty}^{-x-z} f_{\eta'}(n) dn f_{\vartheta'}(z) dz \\
&= 1 - \int_{-\infty}^{+\infty} F_{\eta'}\left(-(x+z)\right) f_{\vartheta'}(z) dz
\end{aligned} \tag{3.35}$$

Since the random variables η' and ϑ' are independent of each other and

$$F_{X'}(x) = \int_{-\infty}^{+\infty} F_{\eta'}\big(-(x+z)\big) f_{\vartheta'}(z)dz$$

where F_{X_i} is the cumulative distribution function of X_i ; $F_{X'}$ is the cumulative distribution function of X' ; $F_{\eta'}$ is the cumulative distribution function of η'; $f_{\eta'}$ is the probability density function of η' ; $f_{\vartheta'}$ is the probability density function of ϑ'.

Theorem 3.2

For a special case of :

a. Equal inventory holding cost and equal penalty cost.

b. The number of different components required to assemble a product is the same for all products.

c. The variance of the demand per period is the same for all products.

d. The sum of the row for the variance-covariance matrix is the same for all components.

the optimal component order-up-to level is

$$Y_j^* = (L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + Q^* \qquad \forall j \tag{3.36}$$

where Q^* is obtained by solving equation (3.37)

$$\int_{-\infty}^{+\infty} F_\eta\left(\frac{Q^*}{m} - z\right) f_{\vartheta'}(z)dz = \frac{Ip + (I-J)mh_c}{I(h' + p + mh_c)} \tag{3.37}$$

<u>Proof</u>

For equal inventory holding cost and equal penalty cost, equation (3.31) can be rewritten as

$$\sum_i \left(h^{'} \int_0^{\infty} x f_{X_i}(x)dx + \left(p + \sum_j G_{ij} h_c \right) \int_{-\infty}^{0} (-x) f_{X_i}(x)dx \right) + E\left[\sum_j h_c \left(Y_j - V_j\right)\right]$$

$$= Ih^{'} \int_0^{\infty} x f_{X_i}(x)dx - I\left(p + mh_c\right) \int_{-\infty}^{0} (x) f_{X_i}(x)dx$$

$$+ E[\sum_j h_c \left(Y_j - (L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + (L+l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} - L\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} - \xi_j\right)]$$

$$= Ih^{'} \int_0^{\infty} x f_{X_i}(x)dx - I\left(p + mh_c\right) \int_{-\infty}^{0} (x) f_{X_i}(x)dx + h_c \sum_j \left(Q_j + (l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}\right) \tag{3.38}$$

It can be shown that at the optimal solution, $Q_j = Q\ \forall j$. The details of the proof can be found in Appendix B. Then, equation (3.34) can be simplified to

$$\eta' = \max\left\{ \frac{\xi_j - Q_j}{m}, j = 1,\ldots,J \right\} = \eta - \frac{Q}{m} \tag{3.39}$$

Substituting (3.39) into (3.33),

$$X_i = -X' = -\eta' - \vartheta^{'} = \frac{Q}{m} - \eta - \vartheta^{'} \tag{3.40}$$

Let $X'' = \eta + \vartheta^{'}$. Then $X_i = \dfrac{Q}{m} - X''$. Equation (3.38) becomes

$$Ih^{'} \int_0^{\infty} x f_{X_i}(x)dx - I\left(p + mh_c\right) \int_{-\infty}^{0} (x) f_{X_i}(x)dx + Jh_c Q + \sum_j \left((l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}\right)$$

$$= Ih^{'} \int_{-\infty}^{Q/m} (Q/m - x) f_{X''}(x)dx + I\left(p + mh_c\right) \int_{Q/m}^{\infty} (x - Q/m) f_{X''}(x)dx$$

$$+ Jh_c Q + \sum_j \left((l+1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu}\right) \tag{3.41}$$

where $F_{X''}(x) = \int_{-\infty}^{+\infty} F_\eta\left(x - z\right) f_{\vartheta^{'}}(z)dz$ and

$F_\eta(x) = \int_{-\infty}^{x} \cdots \int_{-\infty}^{x} f(u_j; j = 1,...,J) du_1 \dots du_J$. $f(u_j;\ j = 1,\dots,J)$ is the joint probability density function of random variables u_j , $j = 1,\dots,J$ given in (3.14).

The first derivative of (3.41) with respect to Q is

$$= Ih' \int_{-\infty}^{Q/m} (1/m) f_{X''}(x)dx + I(p + mh_c) \int_{Q/m}^{\infty} (-1/m) f_{X''}(x)dx + Jh_C$$

$$= \frac{Ih'}{m} \int_{-\infty}^{Q/m} f_{X''}(x)dx - \frac{I(p + mh_c)}{m} \int_{Q/m}^{\infty} f_{X''}(x)dx + Jh_C$$

$$= \frac{Ih'}{m} \int_{-\infty}^{Q/m} f_{X''}(x)dx - \frac{I(p + mh_c)}{m} \left(1 - \int_{-\infty}^{Q/m} f_{X''}(x)dx \right) + Jh_C$$

$$= \frac{I(h' + p + mh_c)}{m} \int_{-\infty}^{Q/m} f_{X''}(x)dx - \frac{I(p + mh_c)}{m} + Jh_c$$

By equating it to be zero, the stationary points occur at

$$\frac{I(h' + p + mh_c)}{m} \int_{-\infty}^{Q/m} f_{X''}(x)dx = \frac{I(p + mh_c) - Jmh_c}{m}$$

$$\int_{-\infty}^{Q/m} f_{X''}(x)dx = \frac{Ip + (I - J)mh_c}{I(h' + p + mh_c)} \tag{3.42}$$

The second derivative of (3.41) with respect to Q is

$$\frac{Ih'}{m} f_{X''}(Q/m)dx + \frac{I(p + mh_c)}{m} f_{X''}(Q/m)dx$$

$$= \frac{I(h' + p + mh_c)}{m} f_{X''}(Q/m)dx \quad \geq 0$$

Since $f_{X''}(Q/m)dx \geq 0$

Hence, equation (3.41) is a convex programming problem. Solving (3.42) gives necessary and sufficient conditions for optimality.

Note that (3.41) is a newsvendor problem and hence the optimal solution is

$$F_{X''}\left(\frac{Q^*}{m}\right) = \int_{-\infty}^{Q/m} f_{X''}(x)dx = \frac{Ip + (I-J)mh_c}{I(h' + p + mh_c)} \tag{3.43}$$

Numerical analysis or the approximation method (Clarke, 1961) can be used to find the optimal Q^* by solving equation (3.43). Then, the optimal component order-up-to level is

$$Y_j^* = (L + l + 1)\left(\vec{G}\vec{e}_j\right)^T \vec{\mu} + Q^* \qquad \forall j$$ □

3.5 Summary

We have developed a closed-form expression for the equal fractile allocation policy which allows component-sharing to determine the optimum order-up-to level. It is evidenced through the analysis of the equal fractile allocation policy the benefits of risk-pooling and the loss due to component-matching in a two-echelon ATS system. These effects are illustrated through a comparative study between the equal fractile allocation policy and the pure push policy. Through probabilistic analysis, we have shown that the equal fractile allocation policy becomes dominant when all components can be shared by at least two products with a high service standard. The risk-pooling effect is amplified with the delivery lead time, the safety factor and the degree of commonality. Under a special scenario (see Theorem 3.2), the cost model of equal fractile allocation can be reduced to a newsvendor problem. As an equal fractile allocation policy aims to achieve equal probability of running out of stock, we should consider other component allocation policies that minimize the total cost. These different policies will be addressed in the following chapters.

CHAPTER 4 MYOPIC ALLOCATION POLICY

4.0 Introduction

Instead of using an equal fractile allocation approach which allocates based on safety factor, this chapter presents a component allocation policy that aims to minimize the total cost. The latter policy takes advantage of component-sharing by pooling the delivery lead time demands and allocates based on the system state and the forecasted demands. For this policy, simulation, Infinitesimal Perturbation Analysis (IPA) and steepest descent algorithm are used to find the associated order-up-to levels of components. Its effectiveness is compared with two policies without component-sharing, namely the pure push policy and the two-echelon policy. In the following section, the two-echelon system and the sequence of events occurring at each period are briefly described. Section 4.2 explains the overall objective function of total cost minimization, the formulation of the myopic allocation policy and the steepest descent algorithm. Section 4.3 describes the IPA method that is used for the gradient estimation. Section 4.4 compares and discusses the simulation results.

4.1 System Description

Consider a manufacturer producing I number of products. These products are assembled from a combination of J number of components, which are ordered from a supplier. When the components arrive, they are released into the assembly process. All components required for the assembly of a product must be available before the assembly process starts. L_j denotes the delivery lead time of component j, and l

denotes the assembly lead time. A schematic diagram of the model is shown in Figure 4.1.

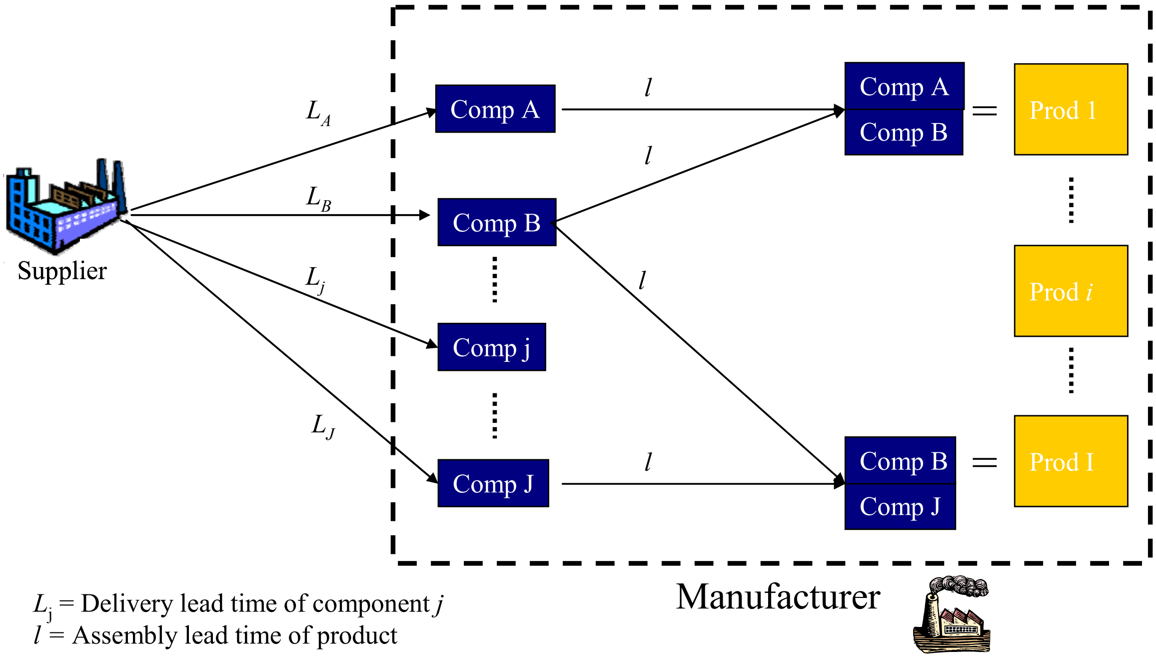


Figure 4.1: A two-echelon supply chain system

A periodic review inventory policy is assumed. To manage the inventory, we track the inventory position in units of component. The inventory position of the component includes the pipeline inventory from the supplier to the manufacturer, the inventory kept at the component level, the inventory in the assembly process, and also the net inventory of the products. Vector $\vec{Y}_t$ denotes the components' order-up-to levels at the beginning of period t after the procurement decision, to allow for demand uncertainty taking account of the delivery lead time, the assembly lead time and the review period. At any period t, orders with quantities $\vec{O}_t$ are placed to bring the inventory positions of components back to $\vec{Y}_t$.

We assume the following:

a. The suppliers' and the manufacturer's production capacities are unlimited.

b. The unfilled demands are back-ordered.

c. The product demands per period are positive, independent and identically distributed.

d. The delivery lead times of the components from the suppliers to the manufacturer are deterministic, but the lead times may vary for different components.

e. The assembly lead time is deterministic.

f. The inventory positions of components are reviewed at every period.

g. L_j and l are multiple integers of the review period for all j.

The following sequence of events occurs each period:

a. At the beginning of period t, orders are placed to replenish the inventory positions of components to $\vec{Y}_t$.

b. The order of component j placed at period t-L_j arrives at the manufacturer's facility.

c. The quantities of components are allocated to products according to the component allocation policy adopted.

d. The quantities of components allocated at period t-l complete the assembly processes.

e. The demand of each product is realized and is met from available product stock. If it is not available, the demand is backlogged.

f. The inventory cost is accrued. There is no fixed order cost. The holding costs are for the excess components and products on-hand at the end of a period, while the back-order costs are on the backlogged demands of products.

4.2. Average Total Cost

A component holding cost of h_{c_j} is charged per unit of component j on-hand. The quantity of component on-hand includes components received but not assigned to any product, and work-in-process components in the assembly line. A product holding cost of h_i is accrued per unit of product i on-hand and a penalty cost of p_i incurred per unit of backlogged demand for product i.

Let $h'_i = h_i - \sum_j G_{ij} h_{c_j}$ and $h_i \geq \sum_j G_{ij} h_{c_j}$. h'_i is the incremental holding cost due to value-added activities. G_{ij} is the element of the matrix $\vec{G}$ which denotes the product structure. G_{ij} is the number of component j used in the assembly of one unit of product i and must be an integer. The average total cost incurred by the manufacturer is denoted by

$$AC = \frac{1}{T} \sum_{t=1}^{T} AC_t \left(\vec{Y}_t \right) \tag{4.1}$$

where $AC_t\left(\vec{Y}_t\right) = \sum_i \left(h'_i \, I_{i,t}^{+} + p_i I_{i,t}^{-} \right) + \sum_j \left(h_{c_j} \left(Y_{j,t} - \sum_i G_{ij} \left(\sum_{k=t-L_j}^{t} d_{i,k} - I_{i,t}^{-} \right) \right) \right)$; Y_j is the jth element of $\vec{Y}_t$ or the order-up-to level of component j at period t; $d_{i,k}$ is the past demand of product i at period k; $I_{i,t}^{+}$ ($I_{i,t}^{-}$) is the amount of inventory on-hand (backlogged demand) for product i at the end of period t; and T is the planning horizon.

From (4.1), the first term $\sum_i \left(h'_i \, I^{+}_{i,t} + p_i I^{-}_{i,t}\right)$ represents the inventory cost incurred by the net inventory of the products. The second term $\sum_j \left(h_{c_j} \left(Y_{j,t} - \sum_i G_{ij} \left(\sum_{k=t-L_j}^{t} d_{i,k} - I^{-}_{i,t} \right)\right)\right)$ represents the holding cost incurred by the components on-hand.

The value of (4.1) is dictated by the component allocation policy adopted and the procurement policy. Both component allocation policy and the procurement policy affect the inventory position of products after the allocation, which in turn affects the value of variable $I_{i,t}$. As product demands are not correlated over time, constant order-up-to levels of components are used as the procurement policy, $\vec{Y} = \vec{Y}_t \;\; \forall t$. Hence, we propose a component allocation policy that minimizes the conditional expectation of total cost for a future period when those newly allocated components complete the assembly process, given $\vec{Y}$ and the latest information. We name this policy as myopic allocation policy. Then, we develop a simulation based algorithm to find $\vec{Y}$ that minimizes the average total cost given in (4.1).

In the bid to find the optimal order-up-to levels using the simulation technique, both component allocation policy and the procurement policy are solved separately. The component allocation model is solved at every simulation period to find the optimal allocation quantities at individual period. We only look at the procurement decision at the end of the simulation run, in which the by-products of the component allocation model or the Lagrange Multipliers are gathered to estimate the gradient. Thereafter, the gradient estimation is used to verify whether the optimal order-up-to levels have been achieved. The optimal order-up-to levels of components are determined interactively based on the gradient estimation of each simulation run. Even though both models are solved separately, the decisions are dependent on each other. The decision on how many components to be allocated to each product depends on the order-up-to level of components and vice versa.

4.2.1 Component Allocation Policy

The myopic component allocation policy allows the delivery lead times of the components from the supplier to the manufacturers to vary for different components, but the lead times are deterministic. To understand the logic of the myopic allocation policy, we first illustrate the interaction of the procurement decision and the allocation decision that impacts on the inventory on-hand at the respective periods.

We consider the order of component *j* placed by the manufacturer at period t-L_j that brings the inventory position of component *j* back to Y_j. The order quantity is the sum of the realized demands that consume component *j* at period t-L_j-1, $\sum_i G_{ij} d_{i,t-L_j-1}$. This order will arrive at period *t*. At this stage, we need to determine the quantity of components to be allocated to assemble each product. Those components that are unassigned at this period will be brought forward to the next period. The assembly process will be completed at period $t+l$. Therefore, the order-up-to levels at period t-L_j affect the allocation decision made at period *t,* which in turn will influence the total cost of period $t+l$ and after, but not prior to, period $t+l$. The exact analysis of finding the optimal component allocation policy, which considers the total inventory cost of period $t+l$ and thereafter, is intractable. We approach this problem by proposing a myopic allocation policy that considers only the inventory cost for one period at a time, which is at period $t+l$.

X_t denotes the system state before the allocation decision at period *t*. The system state includes the previous allocation quantities, the inventory of products, the quantities of components on-hand and past demand. Given X_t and $\vec{Y}$, we propose a component allocation policy which finds the optimal allocation quantities that minimize the conditional expectation of the total cost at period *t+l,* subject to the component availability constraints and the non-negative allocation constraints. The allocation problem at period *t* is formulated as a non-linear program (NLP):

(P4.1)

$$\min_{\vec{a}_t} AC_{t+l}(\vec{a}_t \mid \mathrm{X}_t, \vec{Y}) \; =$$

$$\min_{\vec{a}_t} E\left[\sum_i \left(h'_i\, I^+_{i,t+l} + p_i I^-_{i,t+l}\right) + \sum_j \left(h_{c_j} \left(Y_j - \sum_i G_{ij} \left(\sum_{k=t-L_j+1}^{t+l} D_{i,k} - I^-_{i,t+l} \right)\right)\right) \mid \mathrm{X}_t, \vec{Y} \right] \tag{4.2}$$

subject to
$$\sum_i G_{ij} \left(\sum_{k=t-L_j}^{t-1} d_{i,k} + s_{i,t} + a_{i,t} \right) \le Y_j \qquad \forall j \tag{4.3}$$

$$a_{i,t} \ge 0 \qquad \forall i \tag{4.4}$$

where $\vec{a}_t$ is the vector of decision variables; $a_{i,t}$ is the allocation quantity for product *i* at period *t*; $D_{i,k}$ is the demand of product *i* at period *k*. For $k < t$, $D_{i,k} = d_{i,k}$ which is the past demand of product *i* at period *k*. For $k \ge t$, $D_{i,k}$ is an unknown and random variable which represents the future demand at period *k*; $s_{i,t}$ is the inventory position of product *i* before the allocation at period *t*, consisting of work-in-process in the assembly process and the net inventory of product *i*,. $s_{i,t} = \sum_{k=t-l}^{t-1} a_{i,k} + I_{i,t-1}$.

Equation (4.2) is the objective function of the allocation problem at period *t*. The objective function is an unbiased estimator of the expected total cost of period *t+l*. The net inventory of product *i* at the end of period *t+l* can be obtained by

$$I_{i,t+l} = s_{i,t} + a_{i,t} - \sum_{k=t}^{t+l} D_{i,k} \qquad \forall i \tag{4.5}$$

Denote $D_{i,t}^{l+1} = \sum_{k=t}^{t+l} D_{i,k}$, and its respective probability density function (p.d.f) and cumulative distribution function (c.d.f.) as $f_{D_i^{l+1}}$ and $F_{D_i^{l+1}}$. Thus, Equation (4.2) can be rewritten as:

$$\min_{\vec{a}_t}\Bigg[\sum_i\Bigg(h'_i\int_0^{s_{i,t}+a_{i,t}}\left(s_{i,t}+a_{i,t}-u_i\right)f_{D_i^{l+1}}(u_i)du_i + p_i\int_{s_{i,t}+a_{i,t}}^{\infty}\left(u_i-s_{i,t}-a_{i,t}\right)f_{D_i^{l+1}}(u_i)du_i$$

$$+\sum_j\Bigg(h_{c_j}\Bigg(Y_j-\sum_i G_{ij}\Bigg(\sum D_{i,t}-\int_{s_{i,t}+a_{i,t}}^{\infty}\left(u_i-s_{i,t}-a_{i,t}\right)f_{D_i^{l+1}}(u_i)du_i\Bigg)\Bigg)\Bigg)\,|\,X_t,\vec{Y}\Bigg] \tag{4.6}$$

$$where\ \sum D_{i,t} = \begin{cases} \sum_{k=t-L_j+l}^{t-1} d_{i,t} + (l+1)\mu_i & \text{if } L_j < l+1 \\ (L_j+1)\mu_i & \text{if } L_j \geq l+1 \end{cases}$$

Equation (4.3) represents the component *j* constraint while Equation (4.4) represents the non-negative allocation constraint. Equation (4.3) ensures that total quantity of the components in the system, including the components inventory that have yet to be released into the assembly line, must be less than or equal to the order-up-to levels of that component. Equation (4.4) makes sure that allocation quantities are positive as it is not economic to disassemble components which are in-process or already become part of finished products.

To deal with full differentiable nonlinear program with inequality constraints, Karush-Kuhn-Tucker (KKT) conditions (Rardin, 1998) of (P4.1) is used.

$$h'_i \int_0^{s_{i,t}+a_{i,t}} f_{D_i^{l+1}}(u_i)du_i - \left(p_i + \sum_j G_{ij} h_{c_j}\right) \int_{s_{i,t}+a_{i,t}}^{\infty} f_{D_i^{l+1}}(u_i)du_i + \sum_j G_{ij}\lambda_{j,t} - \lambda_{a_i,t} = 0 \;\; or$$

$$\left(h'_i + p_i + \sum_j G_{ij} h_{c_j}\right) F_{D_i^{l+1}}(s_{i,t} + a_{i,t}) - p_i + \sum_j G_{ij}\left(\lambda_{j,t} - h_{c_j}\right) - \lambda_{a_i,t} = 0 \qquad \forall i \qquad (4.7)$$

$$Y_j - \sum_i G_{ij}\left(\sum_{k=t-L_j}^{t-1} d_{i,k} + s_{i,t} + a_{i,t}\right) \geq 0 \qquad \forall j \qquad (4.8)$$

$$\lambda_{j,t}\left[Y_j - \sum_i G_{ij}\left(\sum_{k=t-L_j}^{t-1} d_{i,k} + s_{i,t} + a_{i,t}\right)\right] = 0 \qquad \forall j \qquad (4.9)$$

$$a_{i,t} \geq 0 \qquad \forall i \qquad (4.10)$$

$$\lambda_{a_i,t} a_{i,t} = 0 \qquad \forall i \qquad (4.11)$$

$$\lambda_{j,t} \geq 0 \qquad \forall j \qquad (4.12)$$

$$\lambda_{a_i,t} \geq 0 \qquad \forall i \qquad (4.13)$$

where $\lambda_{j,t}$ and $\lambda_{a_i,t}$ are the Lagrange multipliers of the component *j* constraint and the non-negative allocation constraint of product *i* respectively.

The Lagrange multipliers should satisfy the sign restriction as given on page 819 (Rardin, 1998). Equation (4.9) and (4.11) are complementary slackness constraints to tackle inequality scenarios arising from knowing what inequalities are active at a local optimum (i.e. hold as equality). Through complementary slackness conditions, when an inequality is active, we treat it as equality. If it is inactive, we leave the constraint out of consideration and the corresponding Lagrange multiplier value must be equal to zero.

Theorem 4.1

The solution that satisfies the KKT conditions above (Equation (4.7) to (4.13)) gives the globally optimal solution for (P4.1).

Proof

The second derivatives of (4.6) with respect to the allocation quantities are

$$\frac{\partial^2 AC_{t+l}\left(\vec{a}_t \mid \mathrm{X}_t, \vec{Y}\right)}{\partial a_{i,t}^2} = \left(h'_i + p_i + \sum_j G_{ij} h_{c_j}\right) f_{D_i^{l+1}}(s_{i,t} + a_{i,t}) \geq 0 \qquad \forall i \qquad (4.14)$$

$$\frac{\partial^2 AC_{t+l}\left(\vec{a}_t \mid \mathrm{X}_t, \vec{Y}\right)}{\partial a_{i,t} a_{b,t}} = 0 \qquad \forall i \neq b \qquad (4.15)$$

A positive semi-definite diagonal Hessian matrix for the objective function is obtained. Since all constraints are linear, the solution space is a convex set and thus (P4.1) is a convex programming problem.

□

4.2.2 Component Order-Up-To Levels

As the product demands are uncorrelated and stationary over time, constant order-up-to levels of components are used. The problem of finding the optimal $\vec{Y}^*$ that minimizes the average total cost as given in (4.1) for the proposed component allocation policy can be formulated as

(P4.2) $$\min_{\vec{Y}} AC = \min_{\vec{Y}} \left[\frac{1}{T} \sum_{t=1}^{T} AC_t(\vec{Y}) \right]$$

subject to the myopic component allocation policy.

As the objective function of the allocation problem, (P4.1), at period *t-l* is an unbiased estimator of the expected total cost of (P4.2) at period *t*, (P4.2) can be rewritten as

$$\text{(P4.3)}\quad \min_{\vec{Y}}\left[\frac{1}{T}\sum_{t=1}^{T} AC_t(\vec{Y})\right] = \min_{\vec{Y}}\left[\frac{1}{T}\sum_{t=1}^{T}\left[AC_t\left(\vec{a}^{\,*}_{t-l} \mid \mathrm{X}_{t-l}, \vec{Y}\right)\right]\right] \qquad (4.16)$$

subject to the myopic component allocation policy.

Not only is the conditional expectation as given in (4.16) an unbiased estimator of the expected total cost that gives a smaller variability, but it also facilitates the implementation of IPA for the gradient estimation.

For a given $\vec{Y}$, we use simulation to estimate the average total cost as given in (4.16). Through IPA (elaborated in Section 4.3), we estimate the gradient vector of the expected cost with respect to the order-up-to levels of components, $\nabla AC\left(\vec{Y}\right)_{\vec{Y}=\left(\vec{Y}\right)_n}$. Based on this gradient information, we adjust $\vec{Y}$ according to the setting $\left(\vec{Y}\right)_{n+1} = \left(\vec{Y}\right)_n - \alpha_n\left(\nabla AC\left(\vec{Y}\right)_{\vec{Y}=\left(\vec{Y}\right)_n}\right)$ (Hjorteland, 1999; Kushner and Yin, 1997), where n is the number of iterations. The amount of adjustment in $\vec{Y}$ is proportional to the value of the gradient estimations evaluated at the n iteration with the selected coefficient of α_n. α_n is referred to as the step size at the n iteration. The harmonic series, $\alpha_n = \alpha/n$ where α is a constant, is chosen for the step sizes so that the convergence of the solution can be guaranteed (Fu, 2002). After $\vec{Y}$ is adjusted, the simulation model is rerun.

The searching procedure is repeated until the stopping criterion is met. The stopping criterion is that either the value of the gradient estimation is not statistically greater than 0 for all order-up-to levels or when the maximum number of iterations is reached. Hitting the maximum number of iterations implies that the convergence rate is too slow. This can be attributed to an inappropriate choice of α. Therefore, a different value for α is chosen and the searching procedure is repeated until the stopping criterion is met. The simulation steps and the steepest descent algorithm are illustrated in Figure 4.2.

We are unable to prove the convexity of the problem (P4.3). Thus, we repeat the searching procedures developed above on several randomly generated initial points $\vec{Y}$ until there is no further improvement or the stopping criterion is met. With different initial starting points, all points stop at close proximity. However, mathematically, the convexity of the problem (P4.3) cannot be proven. Among different searches for a given setting, the point with the lowest average total cost is selected.

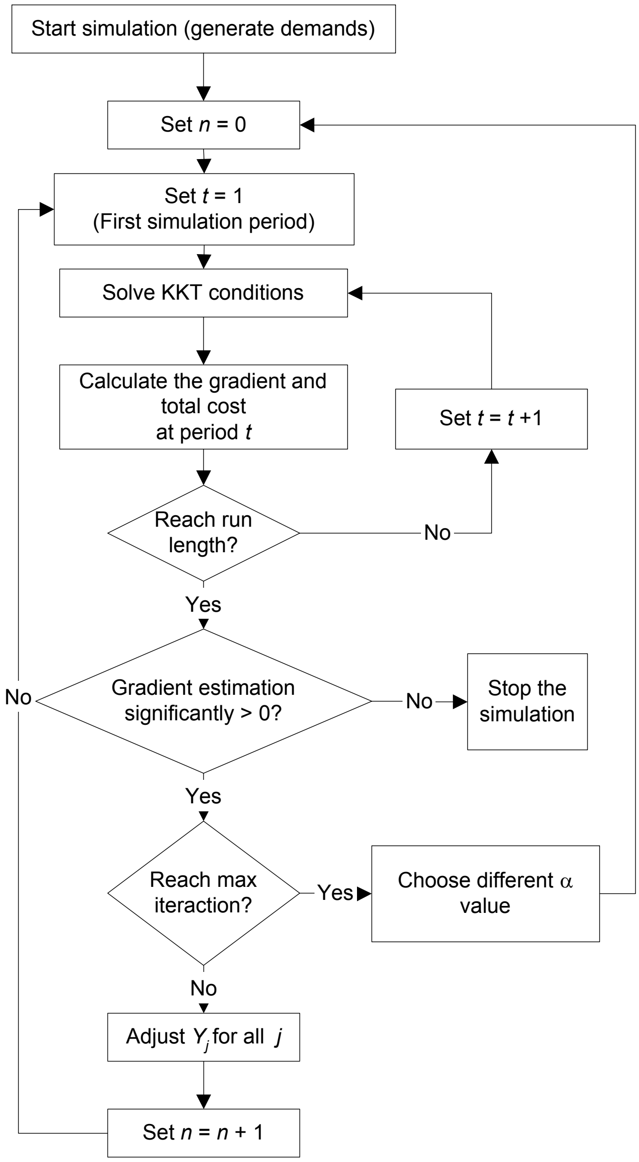


Figure 4.2: Simulation and steepest descent algorithm

4.2.3 Lower Bound

This section develops the lower bound of the average total cost. The proposed component allocation policy is designed basis the system state and the order-up-to levels. We can develop the lower bound by relaxing the conditions. We assume that we are able to determine the system state and the constant order-up-to levels. In other words, we will be able to decide all the components in the pipeline which include components that arrived recently, un-assigned, work-in-process and become part of the finished goods. Since we can decide the system state and the order-up-to levels, the constraints on the component availability and the non-negativity allocation will become void. Hence, the relaxed or unconditional component allocation model becomes:

(P4.4)

$$\min_{\vec{a}_t, \mathrm{X}_t, \vec{Y}} AC_{t+l}(\vec{a}_t) \quad = $$

$$\min_{\vec{a}_t, \mathrm{X}_t, \vec{Y}} E\left[\sum_i \left(h'_i\, I^+_{i,t+l} + p_i I^-_{i,t+l}\right) + \sum_j \left(h_{c_j}\left(Y_j - \sum_i G_{ij}\left(\sum_{k=t-L_j+1}^{t+l} D_{i,k} - I^-_{i,t+l}\right)\right)\right)\right]$$

The unconditional model will yield the same total cost for every period and thereby the total cost at any period will be the same as the average total cost.

4.3 Gradient Estimation

Perturbation analysis has been used in inventory control (Fu 1994). IPA is a technique to evaluate the sample derivatives from a single sample path (Cassandras, 1993; Ho and Cao, 1991; Glasserman, 1991; Fu and Hu, 1997). It is applied to estimate the gradient of Y_j on (4.16). The steps in estimating the gradient through IPA are as follows. We perturb the order-up-to level of component 1 for one period by a small amount $\Delta\vec{Y}$ and investigate how this perturbation affected the corresponding conditional expectation of the total cost and the allocation quantities. We also analyze how the perturbation effect propagated to the subsequent periods. Then, we repeat the perturbation of order-up-to level of component 1 for all periods and capture the corresponding perturbation effect. For the IPA, we take the limit (close to 0) of the change in the perturbation $\Delta\vec{Y}$ and sum up all the perturbation generation and propagation effects to estimate the gradient of Y_1.

The nominal path is defined as the sample path generated by the simulation model with parameter $\vec{Y}$ and the perturbed path as the sample path generated using the same model and the same random seeds, but with parameter $\vec{Y}^{'}$, where $\vec{Y}^{'} = \vec{Y} + \Delta\vec{Y}$. $a_{i,t}^{*}$ and $a_{i,t}^{*'}$ denote the optimal allocation quantity of product i at period t for the nominal path and the perturbed path respectively, where

$$a_{i,t}^{*'} = a_{i,t}^{*} + \Delta a_{i,t} \qquad \forall i,t \qquad (4.17)$$

and $\Delta a_{i,t}$ is the change in the optimal allocation quantity of product i at period t.

4.3.1 Perturbation Generation

To facilitate the derivation of perturbation analysis, we add a subscript t to the order-up-to level of component to indicate the time period when the order is placed or when the inventory position of component is brought up to that level after ordering. We perturb $Y_{1,t-L_1}$ by an infinitesimal small $\Delta Y_{1,t-L_1}$ at period t-L_1 where $\Delta Y_{1,t-L_1}$ is unrestricted in sign. Hence,

$$Y_{j,k}^{'} = \begin{cases} Y_{j,k} + \Delta Y_{j,k} & \text{if } k = t\text{-}L_1 \text{ and } j = 1 \\ Y_{j,k} & \text{others} \end{cases}$$

This perturbation changes the order quantity of component 1 by $\Delta Y_{1,t-L_1}$ at period t-L_1. As this order will arrive at period t, the perturbation effect of $\Delta Y_{1,t-L_1}$ could affect the allocation decision at period t. Both nominal and perturbed path will have the same initial system state, that is $X_t = X_t^{'}$. The perturbation effect on the optimal values of the objective function and the decisions variables of (P4.1) at period *t* depend on the status of component 1 constraint at the optimal solution, whether component 1 constraint is active (binding) or not. Hence, only two scenarios are possible.

Scenario 1: The component 1 constraint is not binding or inactive ($\lambda_{1,t}^{*}$ = 0). The quantity of component 1 is in surplus in the nominal path at period *t*. The change of $Y_{1,t-L_1}$ only affects the quantity of component 1 inventory kept at the component level, and does not affect the optimal allocation quantities. Hence, it implies that $a_{i,t}^{'*} = a_{i,t}^{*}$ $\forall i$ and the change in the objective function value is:

$$\Delta AC_{t+l}(\vec{a}_t \mid X_t, \vec{Y}_{t-L_1}) = AC_{t+l}(\vec{a}_t^{'*} \mid X_t^{'}, \vec{Y}_{t-L_1}^{'}) - AC_{t+l}(\vec{a}_t^{*} \mid X_t, \vec{Y}_{t-L_1})$$

$$= \Delta Y_{1,t-L_j}(h_{c_1}) \tag{4.18}$$

Consequently, the perturbation will affect the component holding cost, as the change in the order quantity affects the quantity of components that are unassigned after the allocation decision at period *t* as there are surplus of component 1. As there are sufficient of component 1, the perturbation of $\Delta Y_{1,t-L_1}$ does not change the optimal allocation quantity.

Scenario 2: The component 1 constraint is binding ($\lambda_{1,t}^{*}$ > 0). The quantity of component 1 is scarce or the reduced cost of the component 1 constraint is positive. More quantity of component 1 available for allocation would reduce the conditional expectation of total cost. Therefore, the perturbation could affect the optimal allocation quantities which lead to $\Delta a_{i,t} \neq 0$ for some *i*. The sum of the change in the allocation quantity which is a subset of component 1 must be equal to the magnitude of perturbation, i.e. $\Delta Y_{1,t-L_1}$. Whereas, the sum of the change in the allocation quantity which is a subset of any other binding component constraints must be equal to zero because there is no perturbation on the order-up-to levels of these components. The quantity of these components available for allocation is not perturbed. The above can be explained mathematically by

$$\sum_i G_{ij} \Delta a_{i,t} = \begin{cases} \Delta Y_{j,t-L_j} & \text{if } j = 1 \\ 0 & \text{Other binding component constraints} \end{cases} \tag{4.19}$$

The change in the optimal objective function value at period *t* can be estimated by

$$\Delta AC_{t+l}(\vec{a}_t \mid \mathrm{X}_t, \vec{Y}_{t-L_1}) \approx \Delta Y_{1,t-L_1}(h_{c_1} - \lambda^*_{1,t}) \tag{4.20}$$

Note that the value of $\Delta Y_{1,t-L_1}$ is assumed to be infinitesimally small (Ho and Cao, 1991) so that the status of all constraints remains unchanged in the perturbed path and so does $\Delta a_{i,t}$ (refer to Equation (4.19)). In our constrained NLP with continuous variables, the probability of having degenerate optimal solutions (one or more binding constraints being redundant) or multiple optimal solutions is close to 0. This validates the assumption stated earlier, that the status of all constraints remains unchanged.

4.3.2 Perturbation Propagation

We have shown how the perturbation of a small quantity of $\Delta Y_{1,t-L_1}$ on $Y_{1,t-L_1}$ affects $\Delta a_{i,t}$ provided the component 1 constraint is binding at period *t*. In this Section, we analyze the propagation effect of $\Delta Y_{1,t-L_1}$ by investigating the effect of $\Delta a_{i,t}$ $\forall i$ on the allocation decision at period t+1 .($a_{i,t+1}$ $\forall i$).

If Scenario 1 happened at period t, there is no change in the optimal allocation quantity at period t, which is $\Delta a_{i,t} = 0$ $\forall i$. Both nominal path and perturbed path will have the same initial system state at period t+1, $X_{t+1} = X'_{t+1}$. Given the same Right-Hand-Side (RHS) of all constraints for the nominal path and the perturbed path $\vec{Y}'_{t-L_1+1} = \vec{Y}_{t-L_1+1}$, this results in the allocation models of both paths arriving at the same optimal solution point, the same inventory positions of products after the allocation, and thus the same objective function value is achieved. There is no propagation effect. Therefore,

$$\begin{aligned}\Delta AC_{t+l+1}(\vec{a}_{t+1} \mid X_{t+1}, \vec{Y}_{t-L_1+1}) &= AC_{t+l+1}(\vec{a}^{'*}_{t+1} \mid X'_{t+1}, \vec{Y}'_{t-L_1+1}) - AC_{t+l+1}(\vec{a}^{*}_{t+1} \mid X_{t+1}, \vec{Y}_{t-L_1+1}) \\ &= 0\end{aligned}$$

In other words, the perturbation effect stops at period t without propagating further.

If Scenario 2 happened at period t, there is a change in the optimal allocation quantity of some products at period t where $\Delta a_{i,t} \neq 0$ for some i. As the objective function of (P4.1) depends on the inventory positions of products after the allocation, the propagation effect depends on whether the effect of $\Delta a_{i,t}$ $\forall i$ can be offset to attain the same inventory positions as per the nominal path at period t+1. In turn, the question of whether the same inventory positions can be achieved depends on the status of the respective non-negative allocation constraints at period t+1. Again, there are two possible scenarios.

Scenario 2.a: $\lambda^{*}_{a_i,t+1} = 0$ ($a^{*}_{i,t+1} > 0$) $\forall\, i \in \{i : \Delta a_{i,t} \neq 0\}$. In this case, the respective optimal allocation quantity at the nominal path is positive at period t+1. Given the same set of objective function, constraints and RHS, the respective optimal allocation quantity at the perturbed path can be 'adjusted' to yield the same solution point. We are able to attain the same inventory position of products after the allocation by having

$$\Delta a_{i,t+1} = -\Delta a_{i,t} \qquad \forall i \qquad (4.21)$$

Hence, $\Delta AC_{t+l+1}(\vec{a}_{t+1} \mid X_{t+1}, \vec{Y}) = 0$.

Scenario 2.b: $\lambda^*_{a_i,t+1} > 0$ ($a^*_{i,t+1} = 0$) for any $i \in \{i : \Delta a_{i,t} \neq 0\}$. In this case, the respective optimal allocation quantity at the nominal path is zero at period t+1. It is more cost-effective not to commit to build this product at this period. In other words, relaxing the non-negative allocation quantity will result in a negative allocation of product *i* and, therefore, lower inventory position of product *i*.

Therefore, under scenario 2.b, $a'^*_{i,t+1}$ and $\Delta a_{i,t+1}$ have to be zero $\forall\, i \in \{i : \lambda^*_{a_i,t+1} > 0$ and $\Delta a_{i,t} \neq 0\}$ for the following reasons:

- If $\Delta a_{i,t} > 0$, the inventory position of product *i* before the allocation of the perturbed path is higher than that of nominal path. The inventory position of product *i* after the allocation cannot be brought down without violating the non-negative allocation constraint since $a^*_{i,t+1} = 0$. The perturbed path is estimated to have a higher total cost of $\left(\lambda^*_{a_i,t+1} \Delta a_{i,t}\right)$.
- If $\Delta a_{i,t} < 0$, the inventory position of product *i* before the allocation of the perturbed path is lower and this component set of $\Delta a_{i,t}$ is available to be released into the assembly line to make product *i*. Nonetheless, $\lambda^*_{a_i,t+1} > 0$

implies that the inventory position of product i is 'enough' to cover lead time demands even before the allocation and thereby it is more cost-effective to allocate components to other products. To ensure that all binding component constraints remain binding in the perturbed path, we have $\sum_i G_{ij}\Delta a_{i,t} + \sum_i G_{ij}\Delta a_{i,t+1} = 0$ if component j constraint is binding *where* $\Delta a_{i,t+1} = 0$ for $\forall\ i \in \{i : \lambda^*_{a_i,t+1} > 0$ and $\Delta a_{i,t} \neq 0\}$. The perturbed path is expected to have a lower total cost of $\left(\lambda^*_{a_i,t+1}\Delta a_{i,t}\right)$ because the component set of $\Delta a_{i,t}$ is available to be used by other products to reduce the total cost.

The above scenarios lead to an estimated change in the optimal objective function value at period t+1 by $\sum_i\left(\lambda^*_{a_i,t+1}\Delta a_{i,t}\right)$. As $\lambda^*_{a_i,t+1} > 0$, the perturbed path will have a higher cost if $\Delta a_{i,t} > 0$ and a lower cost if $\Delta a_{i,t} < 0$. Hence,

$$\Delta AC_{t+l+1}(\vec{a}_{t+1} \mid \mathrm{X}_{t+1}, \vec{Y}) \approx \sum_i\left(\lambda^*_{a_i,t+1}\Delta a_{i,t}\right) \tag{4.22}$$

Similar to as has been described above, the propagation effect may go beyond two periods. But, the probability of having zero allocation quantities for a few consecutive periods is small and, therefore, is ignored.

4.3.3 Total Perturbation

From (4.18), (4.20) and (4.22), the effect of $\Delta Y_{1,t-L_1}$ on the total cost is given by

$$TP_{t-L_1} = \Delta Y_{1,t-L_1}(h_{c_1} - \lambda^*_{1,t}) + \sum_i\left(\lambda^*_{a_i,t+1}\Delta a_{i,t}\right) \tag{4.23}$$

Using the approach described in the previous section, we can estimate the gradient by summing up the perturbation generation and propagation effect of every period

$$\frac{\partial AC(\vec{Y})}{\partial Y_1} \approx \frac{1}{T}\sum_{k=1}^{T}\left(\lim_{\Delta Y_{1,k-L_1}\to 0}\frac{TP_{t-L_1}}{\Delta Y_{1,t-L_1}}\right)$$

$$= \frac{1}{T}\sum_{k=1}^{T}\left((h_{c_1}-\lambda_{1,k}^{*})+\left(\sum_i \lambda_{a_i,k+1}^{*}\left(\frac{\partial a_{i,k}}{\partial Y_{1,k-L_1}}\right)\right)\right) \tag{4.24}$$

An approximation method is developed in Appendix C to estimate the value of $\partial a_{i,t} / \partial Y_{1,t-L_1}$ from the following formula

$$\frac{\partial a_{i1,t}}{\partial Y_{j1,t-L_j}} = \lim_{\Delta Y_{j1,t-L_j}\to 0}\frac{\Delta a_{i1,t}}{\Delta Y_{j1,t-L_j}}$$

In a more general form, we have

$$\frac{\partial AC(\vec{Y})}{\partial Y_j} \approx \frac{1}{T}\sum_{k=1}^{T}\left((h_{c_j}-\lambda_{j,k}^{*})+\left(\sum_i \lambda_{a_i,k+1}^{*}\left(\frac{\partial a_{i,k}}{\partial Y_{j,k-L_j}}\right)\right)\right) \qquad \forall j \tag{4.25}$$

4.4 Results and Discussion

We introduce two other allocation policies, the pure push and the two-echelon, for comparison to measure the performance of the proposed myopic allocation policy. For this comparison, we assume $L_j = L$ $\forall j$. Both policies do not allow component-sharing. The components are allocated to respective products at the time of placing the order and, therefore, the components procured for a product are allocated to that product alone. Without component-sharing, the optimal order-up-to level of product i, S_i^*, can be solved independently with respect to each product, and the order-up-to-level for component j, Y_j^* can be obtained by

$$Y_j^* = \sum_i G_{ij} S_i^* \qquad \forall j \qquad (4.26)$$

Next, we explain the methodology in getting S_i^* for both policies.

Under the pure push policy, only product inventory is kept or stored. The objective function at period *t* aims to minimize the expected total cost of period *t+L+l*,

$$\min_{S_i} E\left[h'_i I^+_{i,t+L+l} + p_i I^-_{i,t+L+l} + \sum_j h_{c_j} G_{ij} \left(S_i - \sum_{k=t}^{t+l} D_{i,k} + I^-_{i,t+L+l} \right) \right] \qquad (4.27)$$

Let us denote $D_i^{L+l+1} = \sum_{k=t}^{t+L+l} D_{i,k}$ with probability density function of $f_{D_i^{L+l+1}}$ and cumulative distribution function of $F_{D_i^{L+l+1}}$, and $D_i^{L+1} = \sum_{k=t}^{t+L} D_{i,k}$ with probability density function of $f_{D_i^{L+1}}$ and cumulative distribution function of $F_{D_i^{L+1}}$. The objective function of (4.27) becomes

$$h'_i \int_{-\infty}^{S_i} (S_i - u_i) f_{D_i^{L+l+1}}(u_i) du_i + \left(p_i + \sum_i G_{ij} h_{c_j} \right) \int_{S_i}^{\infty} (u_i - S_i) f_{D_i^{L+l+1}}(u_i) du_i +$$

$$\sum_j h_{c_j} G_{ij} \left(\int_{-\infty}^{S_i} (S_i - v_i) f_{D_i^{L+1}}(v_i) dv_i \right) \qquad (4.28)$$

Take the partial derivative of Equation (4.28) with respect to S_i and equal it to 0 to obtain the stationary point.

$$h'_i \int_{-\infty}^{S_i} f_{D_i^{L+l+1}}(u_i) du_i - \left(p_i + \sum_i G_{ij} h_{c_j} \right) \int_{S_i}^{\infty} f_{D_i^{L+l+1}}(u_i) du_i + \sum_j G_{ij} h_{c_j} \left(\int_{-\infty}^{S_i} f_{D_i^{L+1}}(v_i) dv_i \right) = 0$$

$$\left(h'_i + p_i + \sum_i G_{ij} h_{c_j}\right)\int_{-\infty}^{S_i} f_{D_i^{L+l+1}}(u_i)du_i - \left(p_i + \sum_i G_{ij} h_{c_j}\right) + \sum_j G_{ij} h_{c_j}\left(\int_{-\infty}^{S_i} f_{D_i^{L+1}}(v_i)dv_i\right) = 0$$

$$F_{D_i^{L+l+1}}(S_i) = \frac{\left(p_i + \sum_i G_{ij} h_{c_j}\right) - \sum_j G_{ij} h_{c_j}\left(F_{D_i^{L+1}}(S_i)\right)}{\left(h'_i + p_i + \sum_i G_{ij} h_{c_j}\right)} \tag{4.29}$$

Take the second derivative with respect to S_i

$$\left(h'_i + p_i + \sum_i G_{ij} h_{c_j}\right) f_{D_i^{L+l+1}}(S_i) + \sum_j G_{ij} h_{c_j}\left(f_{D_i^{L+1}}(S_i)\right) \geq 0 \tag{4.30}$$

Hence, it is obvious that Equation (4.27) is convex because the second derivative (4.30) is non-negative.

$F_{D_i^{L+l+1}}(S_i) \leq F_{D_i^{L+1}}(S_i)$ as l is positive. $F_{D_i^{L+1}}(S_i)$ is likely to be 1 for high service standard requirement. We can find S_i^* by solving Equation (4.31) repeatedly to an accuracy of several decimal points:

$$\Pr(D_i^{L+l+1} \leq (S_i)_{n+1}) = \frac{\left(p_i + \sum_i G_{ij} h_{c_j}\right) - \sum_j G_{ij} h_{c_j}\left(\Pr(D_i^{L+1} \leq (S_i)_n\right)}{\left(h'_i + p_i + \sum_i G_{ij} h_{c_j}\right)} \quad \forall i \tag{4.31}$$

where $(S_i)_n$ is the order-up-to level of product i at n number of iterations. $(S_i)_0$ is set to be ∞.

Under the two-echelon policy, the manufacturer treats the components required to assemble one unit of product as one ‘unit’. The allocation decision is whether to release this ‘unit’ to the assembly line or to store it, without considering sharing with other products. The allocation problem at period t for product i is

$$\min_{a_{i,t}} E\left[h'_i I^+_{i,t+l} + p_i I^-_{i,t+l} + \sum_j G_{ij} h_{c_j} \left(S_i - \sum D_{i,t} + I^-_{i,t+l} \right) | \mathrm{X}_t, S_i \right] \tag{4.32}$$

where $\sum D_{i,t} = \begin{cases} \sum_{k=t-L+l}^{t-1} d_{i,t} + (l+1)\mu_i & \text{if } L < l+1 \\ (L+1)\mu_i & \text{if } L \geq l+1 \end{cases}$

subject to $\sum_{k=t-L}^{t-1} d_{i,k} + s_{i,t} + a_{i,t} \leq S_i$ (4.33)

$a_{i,t} \geq 0$ (4.34)

Similarly, this allocation model is also a convex programming problem.

As there is no closed-form solution for finding S_i^*, similar procedures to those described in the earlier sections, such as simulation, IPA and steepest descent algorithm are applied to search for S_i^*.

By comparing these policies, we are able to quantify the benefits of the echelon effect, of allowing components to be stored as inventory without releasing them into the assembly line, and of the component-sharing effect or component commonality effect. A batch-means simulation technique, with 30 batches each of 1000 periods, is employed to calculate the expected mean and variance of total cost as in (4.16). This has addressed one of the major pitfalls in simulation study as highlighted by Law and McGomas (1986), i.e. conclusions based on a single run as the simulation output are stochastic. The initial data of 100 periods is discarded to allow the simulation system to reach the steady state performance, which is also known as the warm-up period (Welch's procedure suggested by Law and Kelton, 1991).

While our model formulation can be applied to a model with arbitrary number of components and products, we use a small example size to illustrate some key managerial implications. The test case consists of two common components and three products, as shown in Figure 4.3.

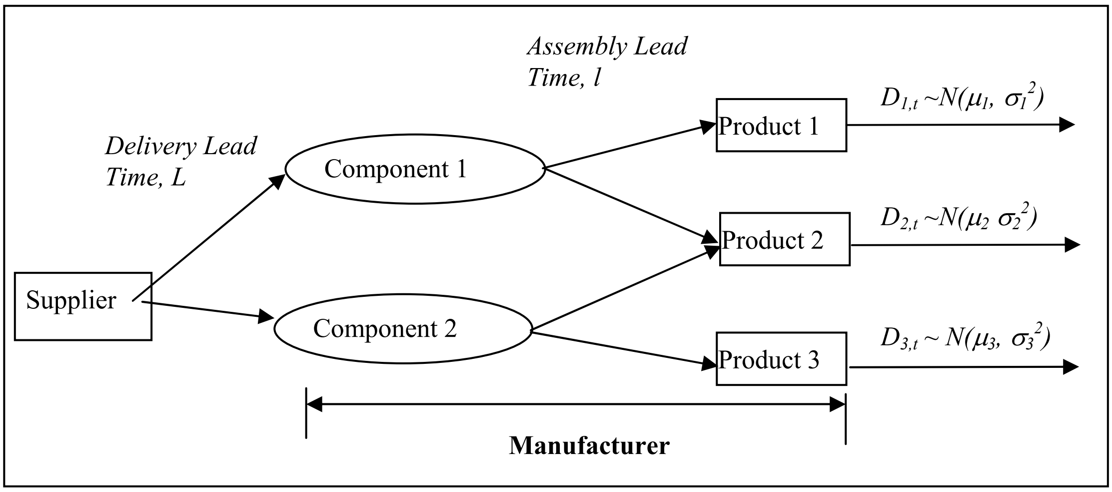


Figure 4.3: A two-component, three-product scenario

The performance measures are average total cost and percentage reduction in the average total cost, taking the pure push policy as the base for comparison. Simulation runs are conducted by varying only one parameter in every experimental setting to assess its impact on the performance measure. The experimental setting used for the comparison is summarized in Table 4.1 with bold letters for the base case setting.

Table 4.1: Experimental setting for uncorrelated demands

Parameters under Study	*Setting*
Delivery lead time: L	1, 3, **5**, 7, 9
Assembly lead time: l	**1**, 3, 5, 7, 9
Incremental holding cost: $h_i^{'} = h^{'}\ \forall i$	**1**, 2, 3, 4
Product penalty cost: $p_i = p\ \forall i$	**10**, 20, 30, 40, 50
Component holding cost: $h_{c_j} = h_c\ \forall j$	**1**,2,3,4,
Demand – Truncated normal with mean $\mu_i = \mu\ \forall i$	**50**
- Standard deviation $\sigma_i = \sigma\ \forall i$	6, 8, **10**, 12, 14

Truncated normal distribution is used for the demand distribution, where any negative random number generated as product demand is discarded. The maximum number of iterations is 1000. Assuming only one unit of component j is used, G_{ij} is 1 when product i uses component j and 0 otherwise.

As the normal distribution function is used for product demands, the expected inventory on-hand at the end of period $t+l$ in (4.6) can be simplified to standardized format (Rogers and Tsubakitani, 1991). This standardized format facilitates the cost calculation during the simulation. The standard format is

$$\int_{-\infty}^{s_{i,t}+a_{i,t}} (s_{i,t} + a_{i,t} - u_i) f_{D_i^{l+1}}(u_i) du_i = \sqrt{\sigma_i^2 (l+1)} \left(Z_{i,t} + R(Z_{i,t}) \right) \tag{4.35}$$

where $\sqrt{\sigma_i^2 (l+1)}$ is the standard deviation of the lead time demand of product i for l+1 periods. $Z_{i,t}$ is the standardized inventory position at the end of period t, $Z_{i,t} = \dfrac{s_{i,t} + a_{i,t} - \mu_i (l+1)}{\sqrt{\sigma_i^2 (l+1)}}$. $R(Z_{i,t})$ is the right-hand unit normal loss integral, $R(Z_{i,t}) = \displaystyle\int_{Z_{i,t}}^{\infty} \frac{\overline{u}_i - Z_{i,t}}{\sqrt{2\pi}} e^{-\frac{\overline{u}_i^2}{2}} d\overline{u}_i$ and $\overline{u}_i = \dfrac{u_i - \mu_i (l+1)}{\sqrt{\sigma_i^2 (l+1)}}$.

Similarly, the number of back-orders at the end of the lead time is given by

$$\int_{s_{i,t}+a_{i,t}}^{\infty}(u_i - s_{i,t} - a_{i,t})f_{D_i^{l+1}}(u_i)du_i = \sqrt{\sigma_i^2(l+1)}\left(R(Z_{i,t})\right) \qquad (4.36)$$

Detailed derivation and the alternative form of $R\left(Z_{i,t}\right)$ for numerical evaluation are explained in Appendix D.

The results show that the difference between the policies for any of the above experimental settings is statistically significant at 95% confidence level (see Appendix E for all hypothesis tests results). Please note that the overall confidence level is lower according to Bonferroni Inequality (Clark and Yang, 1986). Bonferroni Inequality states that if simultaneous multiple interval estimates are required with an overall confidence coefficient 1- α , one can construct each interval with confidence coefficient (1- α/g) where g is the number of multiple internal estimates. The Bonferroni inequality ensures that the overall confidence coefficient is at least 1-α. Figure 4.4 shows that as L increases, the graphs move up because more safety stock is required to buffer the higher demand variability. More stock incurs a higher cost. The difference between the pure push policy and the two-echelon policy indicates the cost-savings due to the echelon effect, while the difference between the two-echelon policy and the myopic allocation policy highlights the cost-savings due to the component-sharing effect. The graph of the myopic allocation policy is consistently lower than the other policies, and thus has the lowest cost. As the demand variability increases with L, the benefit of the echelon effect and the component-sharing effect becomes more pronounced. The graphs depict that the cost-savings due to the component-sharing effect is almost equivalent to the saving due to the echelon effect.

Figure 4.5 illustrates that the percentage reduction in the average total cost decreases as l increases. The benefits of the echelon effect and the component-sharing effect diminish as l increases. The cost tied up in work-in-process inventory becomes higher for a longer assembly lead time, which makes the component-sharing effect and the echelon effect less influential. Nonetheless, the myopic allocation policy outperforms the other allocation policies. The same conjecture can be made as before, that the cost-savings from the component-sharing effect is almost the same as from the echelon effect.

Figure 4.6 and Figure 4.7 show the relationship between the change in the incremental holding cost and component holding cost against the percentage reduction in the average total cost. The benefit of the echelon effect is magnified as h' increases but reduced as h_c increases. As h' increases, the cost of holding inventory of finished products becomes higher. Moreover, the ratio of holding cost of product over the penalty cost is higher. All this gives additional incentive to store more components rather than store the finished products, and thereby make a substantial saving from the echelon effect. In contrast, as h_c increases, the converse is true. The graphs also reveal that the benefit of component-sharing is slightly reduced as h' or h_c increases.

Figure 4.8 and Figure 4.9 illustrate that the percentage reduction increases with the penalty cost and demand variability. From Figure 8, the percentage reduction is not very much affected by the penalty cost. A somewhat high increment of penalty cost from 10 to 50 only results in an increment in the percentage reduction from 12.3% to

16.2% for the myopic allocation policy and from 5.8% to 8.8% for the two-echelon policy. The rise is merely 3% to 4%. From Figure 4.9, the trend highlights that the demand variability can enhance the benefit of the echelon effect and component-sharing. The saving in component-sharing is on a par with the saving in the echelon effect.

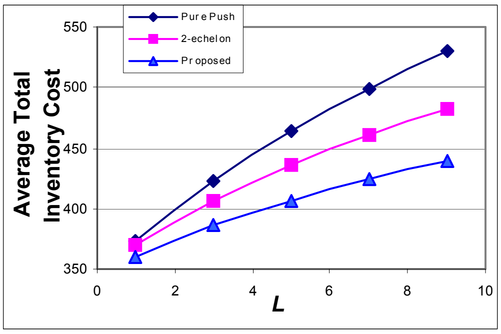


Figure 4.4: Average total cost versus L

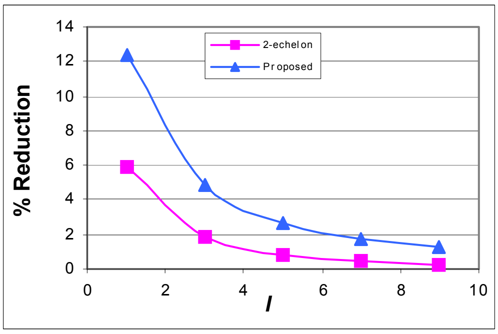


Figure 4.5: % reduction in the average total cost versus l

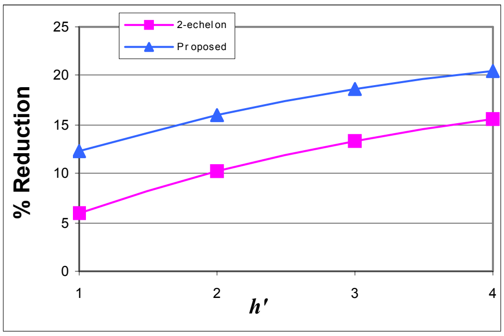


Figure 4.6: % reduction in the average total cost versus h'

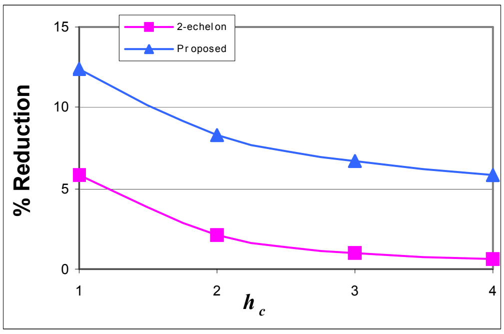


Figure 4.7: % reduction in the average total cost versus h_c

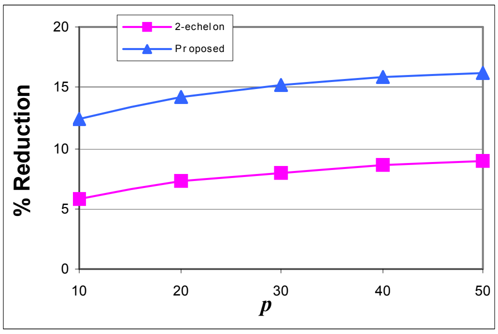


Figure 4.8: % reduction in the average total cost versus p

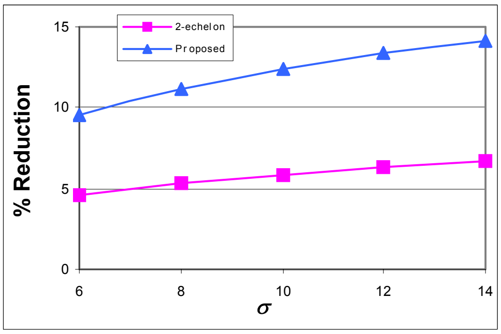


Figure 4.9: % reduction in the average total cost versus σ

We also want to find out how the average total cost obtained by the hill climbing algorithm compared with the initial average total cost found by Eppen and Schrage's method (1981) under different conditions. Note that the same component allocation policy described in Section 4.2 is employed in the simulation to determine the average total costs for both scenarios. Table 4.2 shows the average total cost obtained from the order-up-to-level based on Eppen and Schrage's method and the improvement of the cost after using the proposed method for different p and h' given that $h_c = 1$, $L = 5$, $l = 1$, $\mu = 50$, and $\sigma = 10$. We observe that when both the backlogged cost, p, and the incremental holding cost due to the value-added activities, h', are small, the improvement in cost by the proposed algorithm is small. Hence using Eppen and Schrage's method to generate the order-up-to level yields good initial values. However, when both the p and the h' increase, we observe that the difference in the cost can be as high as 21%. This is because when the incremental holding cost and the backlogged cost are high, it will be better to hold more inventories at the component level.

Table 4.2: Comparison between initial starting points by Eppen and Schrage (1981) and the proposed method

P	h'	Initial Cost by Eppen & Schrage	Local Optimal Cost by Proposed Method	Ratio of Initial Cost /Optimal Cost
10	1	406.1	406.1	1
10	10	691.2	643.7	1.07
10	30	978.9	839.6	1.17
10	50	1130.2	936.6	1.21
30	1	702.2	690.9	1.02
30	10	934.7	887.7	1.05
30	30	1560.0	1412.8	1.10
30	50	1960.5	1624.6	1.21
50	1	764.3	757.2	1.01
50	10	1043.2	1003.4	1.04
50	30	1859.9	1652.3	1.13
50	50	2425.9	2048.3	1.18

4.5 Summary

The two-echelon supply chain model is characterized by long procurement lead times for components which are subsequently used in the assembly of several products in an ATS environment. We have proposed a myopic allocation policy that minimizes the conditional expected total cost for a future period when those allocated components have completed the assembly process and become available to fill the actual demands. The policy allows component-sharing. The allocation model is a convex problem at any given period. We have combined simulation, IPA and steepest descent algorithm to search for the constant order-up-to levels of components for the myopic allocation policy.

This chapter has quantified the benefits of component-sharing and the echelon effect. The results reveal that the proposed policy consistently performs better than the two-echelon policy because of the component-sharing effect. The saving through component-sharing is quite substantial and is almost the same as the saving through the echelon effect, except where there is a high incremental holding cost. Consequently, the myopic allocation policy is highly recommended.

We have assumed that the demands are independent over time. In the next chapter, we will introduce demand correlation into the model and proposes a dynamic procurement policy that considers the component-sharing based on the latest demand information.

CHAPTER 5 DYNAMIC LEVEL PROCUREMENT POLICY

5.0 Introduction

Up to this point, we have assumed independence of the demand for various products over time. However, occasionally that may not be the case in the real world, as demand is likely to be correlated over time. As mentioned by Nelson (1976), when time series are used to forecast the sales of various products in a firm, correlations are likely to occur between the residuals of forecasting errors for different products. Demand correlation may affect the decision on the procurement / order quantity and allocation because of the value of latest information. Based on the latest information available, we can update forecast demands at every review period and use this information to better plan the quantity of components to procure from the suppliers and the quantity to be released into the production line. The demands are modeled by an auto-regressive process to capture the time-correlated demands. The purpose of this chapter is to evaluate the effect of component-sharing and the effect of dynamic order-up-to level over constant order-up-to level on the total costs in the presence of demand correlation. Constant order-up-to level, which implies the order-up-to level is constant over time, assumes independence of demand and is not able to react to change in the forecast demand. Dynamic order-up-to level does take into account the latest demand information and adjusts the order-up-to level accordingly. In other words, this policy allows the order-up-to level to be dynamic or fluctuate over time, and the optimal order quantity is determined at every review period. However, determining the optimal order quantity of components to minimize the average total cost is a stochastic problem that is hard to solve because of the inter-dependency of procurement decision and allocation decision as well as the demand randomness and demand correlation. How much one should order is affected by the allocation decision at future periods as this order quantity will only be available for allocation after a pre-defined delivery lead time. After the order arrival, how one should allocate the order quantity to products is affected by the system state or scenario at that period. A vast number of scenarios are possible because the allocation decision is solved sequentially at every period based on the latest demand information. Hence, the system state after the order arrival depends on the allocation decisions and the demand realized during the lead time of delivery. Due to the intricacies of inter-dependency between the procurement and allocation decisions of various periods and the time-correlated demands, a Sample Average Approximation (SAA) method is used for dynamic order-up-to level model to approximate the total cost in order to capture the change in the forecasted demands which are updated periodically based on the latest available information. SAA will estimate the impact of changes in the forecasted demands on the allocation and procurement decisions to identify the optimal allocation quantity and order quantity that minimize the estimated total cost. For simulation-based optimization approach as discussed in Chapter 4 is presented to determine the order-up-to level for Constant Level with Sharing model.

To evaluate the effects of component-sharing and the effect of dynamic order-up-to level over the constant order-up-to level, we compare the relative performance of three procurement policies, i.e. Constant Level with Sharing, Dynamic Level without Sharing and Dynamic Level with Sharing. Constant Level with Sharing, as the name implies, has the constant order-up-to level. This procurement policy will bring the inventory position of every component back to its original level at every period regardless of the availability of latest demand information. This policy does take advantage of the component commonality in employing myopic allocation policy when determining the optimal order-up-to level. Dynamic Level without Sharing allows the order-up-to levels to be dynamic over time and treats individual demand for product to be independent without considering component-sharing. Dynamic Level with Sharing considers both change in the order-up-to level at every period based on the updated forecast demands and component-sharing. The effect of component-sharing can be obtained by measuring Dynamic Level with Sharing and Dynamic Level without Sharing. The effect of dynamic level over constant level can be obtained by measuring Dynamic Level with Sharing against Constant Level with Sharing.

This chapter is organized as follows. A review of the two-echelon system can be found in Section 5.1. In Section 5.2, the conditional means and variance of multiple period forecast demands is presented. The detailed formulation of the myopic allocation model is explained in Section 5.3; while the three procurement policies for comparison and their model formulation are introduced in Section 5.4. Section 5.5 provides the numerical results to compare the performance of these policies and a summary is presented in the final section.

5.1 System Description

This section recaptures the supplier-manufacturer system under study. Consider a manufacturer producing I number of products. These products are assembled from a combination of J number of components, which are ordered from a supplier. When the components arrive, they are released into the assembly process. All components required for the assembly of a product must be available before the assembly process starts. L denotes the delivery lead time of component, and l denotes the assembly lead time. A schematic diagram of the model is shown in Figure 5.1.

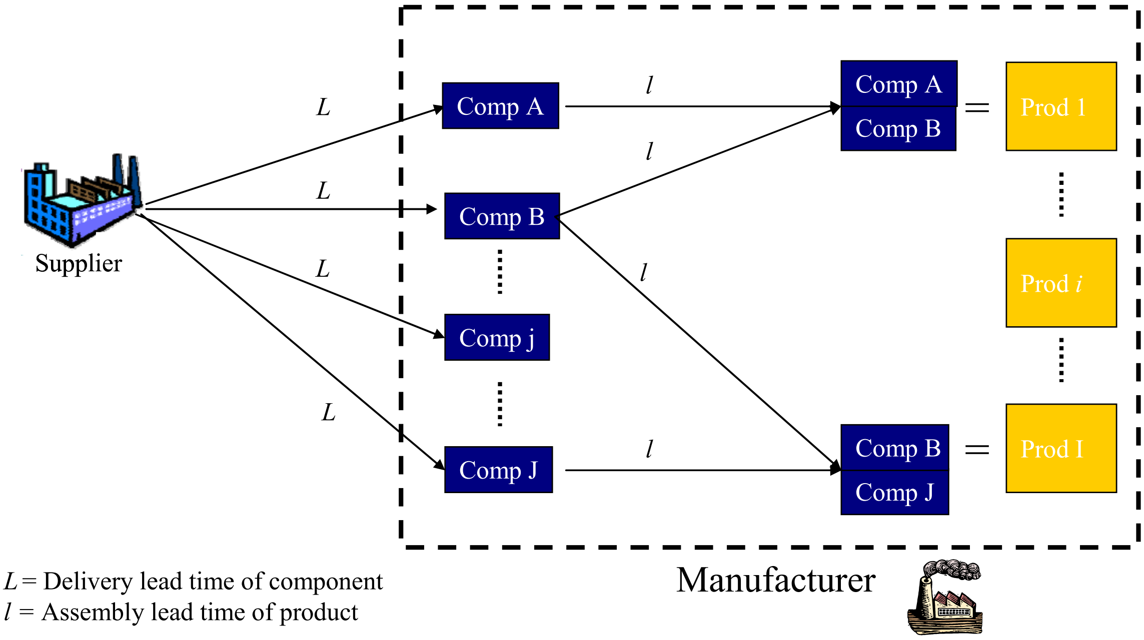


Figure 5.1: A two-echelon supply chain system

As in the previous work, a periodic review policy is assumed for every period. We want to track the inventory position at the component level, which includes the pipeline inventory from the supplier to the manufacturer, the inventory kept at the component level, the inventory in the assembly process, and also the net inventory of the products. Let vector $\vec{Y}_t$ and $\vec{O}_t$ denote the components' order-up-to levels and the order quantities of components at the beginning of period t.

The following assumptions are made:

a. The suppliers' and the manufacturer's production capacities are unlimited.

b. The unfilled demands are back-ordered.

c. The product demands per period are positive and auto-correlated but independent of each other.

d. The delivery lead times and the assembly lead time are deterministic.

e. L and l are multiple integers of the review period.

The following sequence of events occurs at each period as follows:

a. At the beginning of period t, forecasts are updated for the demands to be realized, counting the current period as the first period.

b. With the updated forecast demands in-hand, order $\vec{O}_t$ is placed to replenish the inventory positions of components to $\vec{Y}_t$.

c. The order placed at period t-L ($\vec{O}_{t-L}$) arrives.

d. Quantities of the components are allocated to products according to the component allocation policy adopted.

e. The quantities of components allocated at period t-l complete the assembly processes.

f. The demand of each product materializes and is met from available product stock; the unmet demand is backlogged, which determines the inventory stock or backlog level at the end of the current period. The excess will carry over to the next period.

g. The total cost is accrued. There are no fixed order costs. The holding costs are for the excess components and products on-hand at the end of a period, while the back-order costs are on the backlogged demands of products.

In other words, the order placed at period t will become available for allocation decision at period $t+L$; some quantity of components is unassigned and is kept at component level after the allocation decision at $t+L$. Those components allocated will complete the assembly process at period $t+L+l$ and be made available to fulfill customer demands. Hence, the total cost at period $t+L+l$ is mainly affected by both the procurement decision at period t and the allocation decision at period $t+L$ (see also Figure 5.2).

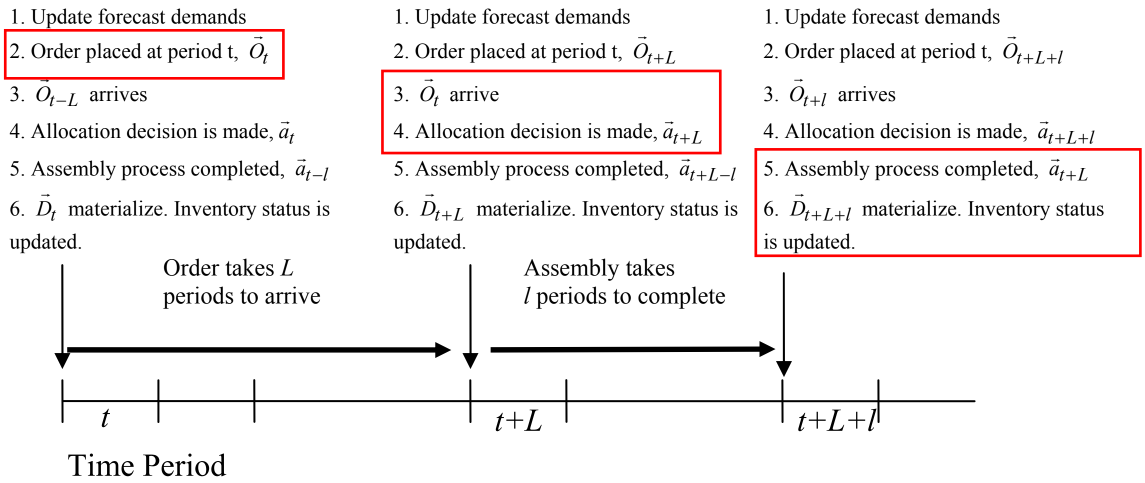


Figure 5.2: A timeline of system dynamic on procurement and allocation decisions

At the end of every period, inventory-holding and back-order-penalty costs are charged. There are linear costs for inventory-holding and backlogging, respectively. We use h_{c_j}, h_i and p_i to represent the inventory-holding of component j on-hand which includes components in inventory and in the assembly process, product-holding cost and back-order-penalty costs, respectively.

Let G_{ij} be the element of the matrix $\vec{G}$ which denotes the product structure. G_{ij} is the quantity of component j used in the assembly of one unit of product i and must be an integer. The average total cost over T periods incurred by the manufacturer can be denoted by

$$AC = \frac{1}{T}\sum_{t=1}^{T} AC_{t+L+l} \tag{5.1}$$

$$= \frac{1}{T}\sum_{t=1}^{T} \left(\sum_i \left(h_i\, I^{+}_{i,t+L+l} + p_i I^{-}_{i,t+L+l} \right) + \sum_j \left(h_{c_j} \left(\sum_i \left(G_{ij} \sum_{k=t+L+1}^{t+L+l} a_{i,k} \right) + \theta_{j,t+L+l} \right) \right) \right) \tag{5.2}$$

where $$\sum_i G_{ij} a_{i,t+L} + \theta_{j,t+L} = O_{j,t} + \theta_{j,t+L-1} \qquad \forall j,\ t \tag{5.3}$$

$$I_{i,t+L+l} = I_{i,t+L+l-1} + a_{i,t+L} - D_{i,t+L+l} \qquad \forall i,\ t \tag{5.4}$$

$O_{j,t}$ is the order quantity of component *j* at period *t* and $\vec{O}_t$ is the vector of order quantities*;* $I^{+}_{i,t+L+l}$ ($I^{-}_{i,t+L+l}$) is the amount of net inventory on-hand (backlogged demand) for product *i* at the end of period *t+L+l*; $a_{i,t+L}$ is the allocation quantity of product *i* at period *t+L*; $\theta_{j,t+L}$ is the quantity of component *j* that is unassigned to any product at the end of period *t+L;* and *T* is the planning horizon that is large enough to give an unbiased estimation of average total cost.

From (5.2), the first term $\sum_i \left(h_i\, I^{+}_{i,t+L+l} + p_i I^{-}_{i,t+L+1} \right)$ represents the inventory cost incurred by the net inventory of the products whereas the second term $\sum_j \left(h_{c_j} \left(\sum_i \left(G_{ij} \sum_{k=t+L+1}^{t+L+l} a_{i,k} \right) + \theta_{j,t+L+l} \right) \right)$ represents the holding cost incurred by the components on-hand while $\sum_i \left(G_{ij} \sum_{k=t+L+1}^{t+L+1} a_{i,k} \right)$ represents the work-in-process to assemble product *i*. The average total cost is the sum of the average total cost at period *t+L+l* and thereafter, where the first order at period *t* has sufficient time to go through the whole delivery and assembly process and to be ready for the demand fulfillment, assuming the system starts as an empty system at period *t*.

Equation (5.3) represents the component availability constraint at period $t+L$. It shows that the sum of the allocation quantity and the unassigned components after the allocation at the end of period $t+L$ (the summation can be denoted by $\sum_i G_{ij} a_{i,t+L} + \theta_{j,t+L}$) must be equal to the quantity of component available for allocation at the beginning of that period which is the sum of the unassigned components carried forward from the previous period and the arrival of order placed at period t (the summation is denoted by $O_{j,t} + \theta_{j,t+L-1}$).

Equation (5.4) is the inventory of product i at period $t+L+l$, which is equivalent to the sum of the inventory of product i at the end of the previous period and the quantity of product that completes the assembly process at that period, with product demand at that period subtracted.

Next, we transform the objective functions (5.2) by using order quantities as parameters instead of allocation quantities. This transformation will facilitate the development of a procurement and allocation model in the subsequent sections. The quantity of component that is unassigned and still in the assembly line is

$$\sum_i G_{ij} \sum_{k=t+L+1}^{t+L+l} a_{i,k} + \theta_{j,t+L+l} = \sum_i G_{ij} \sum_{k=t+L+1}^{t+L+l-1} a_{i,k} + \sum_i G_{ij} a_{i,t+L+l} + \theta_{j,t+L+l} \tag{5.5}$$

From (5.3), we can derive that

$$\sum_i G_{ij} a_{i,t+L+l} + \theta_{j,t+L+l} = O_{j,t+l} + \theta_{j,t+L+l-1} \tag{5.6}$$

By substituting (5.6) into (5.5), we get

$$= \sum_i G_{ij} \sum_{k=t+L+1}^{t+L+l-1} a_{i,k} + O_{i,t+l} + \theta_{j,t+L+l-1} \qquad \forall j$$

By inserting component availability constraint of the respective periods, we have

$$= \sum_{k=t+1}^{t+l} O_{j,k} + \theta_{j,t+L} \qquad \forall j \qquad (5.7)$$

By inserting (5.7) into (5.2), we have

$$AC_{t+L+l} = \sum_i \left(h_i I^{+}_{i,t+L+l} + p_i I^{-}_{i,t+L+l} \right) + \sum_j \left(h_{c_j} \left(\sum_{k=t+1}^{t+l} O_{j,k} + \theta_{j,t+L} \right) \right) \qquad (5.8)$$

By substituting $\theta_{j,t+L} = O_{j,t} + \theta_{j,t+L-1} - \sum_i G_{ij} a_{i,t+L}$ into (5.8), (5.2) becomes

$$\frac{1}{T} \sum_{t=1}^{T} \left(\sum_i \left(h_i I^{+}_{i,t+L+l} + p_i I^{-}_{i,t+L+l} \right) + \sum_j \left(h_{c_j} \left(\sum_{k=t}^{t+l} O_{j,k} + \theta_{j,t+L-1} - \sum_i G_{ij} a_{i,t+L} \right) \right) \right) \qquad (5.9)$$

To obtain the optimal procurement and allocation policy to minimize (5.9), dynamic programming can be tasked to do the job. However, there is an important practical limitation to our ability to solve increasingly detailed and realistic dynamic programming problems, namely the curse of dimensionality (Bellman, 1955). This is the well-known exponential rise in the amount of time and space required to compute the solution to a dynamic programming problem as the number of dimensions of the state variable or of the control variable increases. Furthermore, dynamic programming is hard to solve especially when the state space becomes very large.. In view of all the factors mentioned, instead of looking at minimizing the average total cost, we employ a greedy approach to minimize the total cost at the corresponding period. When discussing the procurement model, our reference period is at period t and the objective is to minimize the total cost at period $t+L+l$. When solving the allocation model, our reference period is at period $t+L$ and the objective is also to minimize the total cost at period $t+L+l$ (see Figure 5.2 for illustration).

We explain the sequence of events occurring at every period. Firstly, we introduce the methodology of updating the forecast of multiple-period demands. Secondly, we explain the myopic allocation policy, and thirdly, we put forward the three procurement policies considered.

5.2 Demand Forecast

At the beginning of every period, we generate forecasts of the demand for future periods in the planning horizon based on past demand. Instead of a single-period forecast, the forecast demand across k periods is used to predict demand over the planning horizon. For the procurement model, the forecast demand across L+l+1 periods is used; while for the allocation model, it is forecast across l+1 periods. Thus, the lead time demand forecast is expressed as an aggregate of multiple forecasts with the forecasting horizon ranging from t to $t+L+l$ for the procurement model and $t+L$ to $t+L+l$ for the allocation model.

The demands are auto-correlated and follow an auto-regressive process, but are independent from each other. The total demand of product i for k periods obtained using the auto-regressive model has a normal distribution with mean $\mu_i(k \mid \vec{d})$ and variance $\sigma_i^2(k \mid \vec{d})$ that is conditional on past demand, denoted by vector $\vec{d}$. The conditional mean and the conditional variance are

$$\mu_i(k \mid \vec{d}) = k\mu_i + \Lambda(\vec{d}, i, k) \qquad \forall i \qquad (5.10)$$

$$\sigma_i^2(k \mid \vec{d}) = \sigma'^2_i \, \Gamma(\vec{d}, i, k) \qquad \forall i \qquad (5.11)$$

where μ_i is the mean demand of product i for one period; σ'^2_i is the variance of the white noise from the auto-regressive process of product i; $\Lambda(\vec{d},i,k)$ is the weighted information of the past demand of product i for k periods; and $\Gamma(\vec{d},i,k)$ is a coefficient to capture the demand correlation of product i across k periods. The detailed derivation of these terms is given in Appendix G.

5.3 Myopic Component Allocation Policy

The procurement policy and the allocation policy that exploit the component commonality in a two-echelon ATS system are considered. This problem requires the determination of the procurement quantity and subsequent allocation quantity to different products. We address the allocation policy first and then determine the order quantity. The allocation policy determines the quantity of components to be released into the assembly process based on system state, order quantity and forecast demand.

As mentioned earlier, when solving the allocation model, our reference period is $t+L$. The objective of the allocation decision is the minimization of the total cost at period $t+L+l$. The following information is needed and available at period $t+L$ before making the allocation decision:

- The previous allocation quantities, $a_{i,k} \quad \forall i,k < t+L$
- The inventory of products at the previous period, $I_{i,t+L-1} \quad \forall i$
- The quantities of unassigned components on-hand at the previous period, $\theta_{j,t+L-1} \quad \forall j$
- The current and previous order quantities, $O_{j,k} \quad \forall j,k \le t+L$

- The past demand, $d_{i,k} \quad \forall i, k < t+L$

The decision variables are the allocation quantities at period *t*+*L*. The by-product is the quantity of unassigned components to be stored for the following periods.

For the model formulation, let X_{t+L} denote the system state before the allocation decision at period *t*+*L*. The system state includes the previous allocation quantities, the inventory of products, the quantities of components on-hand and past demand. Given X_{t+L} and $\vec{O}_t$, the myopic allocation policy finds the optimal allocation quantities that minimize the conditional expectation of the total cost at period *t*+*L*+*l*, subject to the component availability constraints and the non-negative allocation constraints. The allocation problem at period *t*+*L* is formulated as a non-linear program (NLP)

(P5.1)

$$\min_{\vec{a}_{t+L}} AC_{t+L+l}(\vec{a}_{t+L} \mid \mathrm{X}_{t+L}, \vec{O}_t) =$$

$$\min_{\vec{a}_{t+L}} E\left[\sum_i \left(h_i I^+_{i,t+L+l} + p_i I^-_{i,t+L+l}\right) + \sum_j \left(h_{c_j}\left(\sum_{k=t}^{t+l} O_{j,k} + \theta_{j,t+L-1} - \sum_i G_{ij} a_{i,t+L}\right)\right) \mid \mathrm{X}_{t+L}, \vec{O}_t\right] \tag{5.12}$$

subject to
$$O_{j,t} + \theta_{j,t+L-1} = \sum_i G_{ij} a_{i,t+L} + \theta_{j,t+L} \qquad \forall j \tag{5.13}$$

$$a_{i,t+L} \geq 0 \qquad \forall i \tag{5.14}$$

$$\theta_{j,t+L} \geq 0 \qquad \forall j \tag{5.15}$$

where the inventory of product *i* at period *t*+*L*+*l* can be obtained by

$$I_{i,t+L+l} = I_{i,t+L-1} + \sum_{k=t+L-l}^{t+L} a_{i,k} - \sum_{k=t+L}^{t+L+l} D_{i,k} \qquad \forall i \tag{5.16}$$

$\vec{a}_t$ is the vector of decision variables; $a_{i,t}$ is the allocation quantity for product *i* at period t; $O_{j,k}$ is the order quantity of component j at period k. For $k \le t + L$, $O_{j,k}$ is known which is the past order at period k. For $k > t + L$, $O_{j,k}$ is an unknown variable which represents future orders at period k; $D_{i,k}$ is the demand of product i at period k. $\sum_{k=t+L}^{t+L+l} D_{i,k}$ is the sum of the future demand from period $t+L$ through $t+L+l$. It is unknown and follows a normal distribution with the conditional mean and variance given in (5.10) and (5.11).

Equation (5.12) represents the conditional expectation of the total cost at period $t+L+1$; Equation (5.13) represents the component availability constraint. Equation (5.14) and (5.15) are the non-negativity constraint of allocation quantity and quantity of unassigned components.

At period $t+L$, $O_{j,k}$ for $k \ge t + L + 1$ is unknown. These future orders will not impact the optimum solution point except for the objective function value. $O_{j,k}$ for $k \ge t + L + 1$ will incur higher component holding cost if $O_{j,k} > 0$. Hence, $O_{j,k} = 0$ for $k \ge t + L + 1$ when solving (P5.1) to minimize the total cost of period $t+L+l$. The convexity of myopic allocation problem has been provided in Section 4.2.1 which can be extended to (P5.1). Solving the KKT conditions of (P5.1) will yield an optimum solution.

5.4 Procurement Policy

If the demands are stationary and independent over time, the expected mean of the forecast demands does not vary over time and therefore need not be updated based on the latest demand information. When the demands are correlated over time, the mean and variance of the forecast demands are conditional on past demand. Hence, we update the forecast demands at every review period by solving (5.11) and (5.12). In this case, we have a choice whether to keep the order-up-to levels constant or dynamic. If we keep the levels constant, then the order-up-to levels do not change over time. For constant level policy, we adapt the same simulation methodology as discussed in Chapter 4 to find the optimum order-up-to levels. We do not look at the procurement decision at every simulation period. However, we will determine the optimal order-up-to levels based on the gradient estimation at the end of the simulation run to re-adjust the levels if necessary. For dynamic level policy, we will determine the order-up-to levels at every period after the demand forecast is updated and the components are allocated. After the update of demand forecast, we will look at the component allocation model first. After which, we will solve the procurement model to determine order-up-to levels. For example, for the Dynamic Level with Sharing, we solve the component allocation model (P5.1) before solving the procurement model (P5.2).

5.4.1 Dynamic Level with Sharing

When solving the procurement decision at period t, the following information is needed and available at period t before determining the quantity of components to procure.

- The previous allocation quantities, $a_{i,k}$ $\forall i, k < t$
- The inventory of products at the previous period, $I_{i,t-1}$ $\forall i$
- The quantities of unassigned components on-hand at the previous period, $\theta_{j,t-1}$ $\forall j$

- The previous order quantities, $O_{j,k} \quad \forall j,k < t$
- The past demand, $d_{i,k} \quad \forall i,k < t$

The decision variables are the order quantities at period *t.*

The problem of finding the order quantity at period *t* aims to minimize the conditional expectation of total cost at period *t+L+l* which can be formulated as

(P5.2)

$$\min_{\vec{O}_t} AC_{t+L+l}\left(\vec{O}_t \mid \mathrm{X}_t\right) =$$

$$\min_{\vec{O}_t} E\left[\sum_i \left(h_i I^+_{i,t+L+l} + p_i I^-_{i,t+L+l}\right) + \sum_j \left(h_{c_j}\left(\sum_{k=t}^{t+l} O_{j,k} + \theta_{j,t+L-1} - \sum_i G_{ij} a_{i,t+L}\right)\right) \mid \mathrm{X}_t\right] \tag{5.17}$$

subject to

the myopic allocation policy (solving (P5.1) sequentially from period *t* through *t+L*)

$$O_{j,t} \geq 0 \qquad \forall j \tag{5.18}$$

where $I_{i,t+L+l} = I_{i,t-1} + \sum_{k=t-l}^{t+L} a_{i,k} - \sum_{k=t}^{t+L+l} D_{i,k} \quad \forall i$. Equation (5.18) is the non-negative order constraint.

The exact analysis of finding the optimal order quantities to minimize the conditional expectation of total cost given in (5.17) is very hard and intractable due to the complexity in the relationship between the order quantity at period *t* and the allocation decision at period *t+L* (see Figure 5.2 for the graphical representation). Moreover, the allocation decision at period *t+L* depends on the sequential allocation decisions from period *t* through period *t+L*-1 where these allocation solutions have not been decided when making the procurement decision at period *t*. For example, the future allocation decisions at period $t+L$ depend on the allocation quantities, demand realization and received quantities of component at period $t+L$-1, which in turn depends on the allocation quantities, demand realization and received quantities of component at period $t+L$-2, which in turn depends on the previous periods and so forth. Furthermore, the demands are correlated and forecast demands will vary. The forecast demands will be updated every period when making the allocation decisions. Hence, the stochastic elements of having a prohibitively large set of random system states could happen at period $t+L$ before the allocation. The use of exact mathematical programming techniques (for example, the L-shaped method) to estimate the total cost becomes ineffective (Akcay and Xu, 2004). We shall use the Sample Average Approximation (SAA) method to estimate the sample average of the total cost when determining the optimal order quantity.

The SAA method is a Monte Carlo simulation-based solution approach to stochastic optimization problems (Verweij et al., 2003; Royset, 2004). The conditional expectation function of (5.17) cannot be computed exactly, but it can be estimated by Monte Carlo simulation. By randomly generating a sample path ω_n as the sequences of auto-correlated demands for different products $\left\{\vec{D}_t^{(\omega_n)}, \vec{D}_{t+1}^{(\omega_n)}, \vec{D}_{t+2}^{(\omega_n)}, \ldots\ldots\ldots\vec{D}_{t+L-1}^{(\omega_n)}\right\}$ where ω_n represents the n-th sample path, by the Law of Large Numbers, we approximate it by the corresponding sample average function (Kleywegt et al., 2001). Roughly speaking, the SAA method approximates the conditional expectation of the stochastic program with a sample average estimation based on a number of randomly generated sample paths. Firstly, we sequentially solve the allocation decision from period t to period $t+L$-1 for every sample path. Then, in a single model, we simultaneously solve the allocation decisions at period $t+L$ of all sample paths and the procurement decision. This single model determines the order quantity at period t that minimizes the sample average function of the total cost at period $t+L+1$. The detailed process flow for the SAA method is plotted in Figure 5.3.

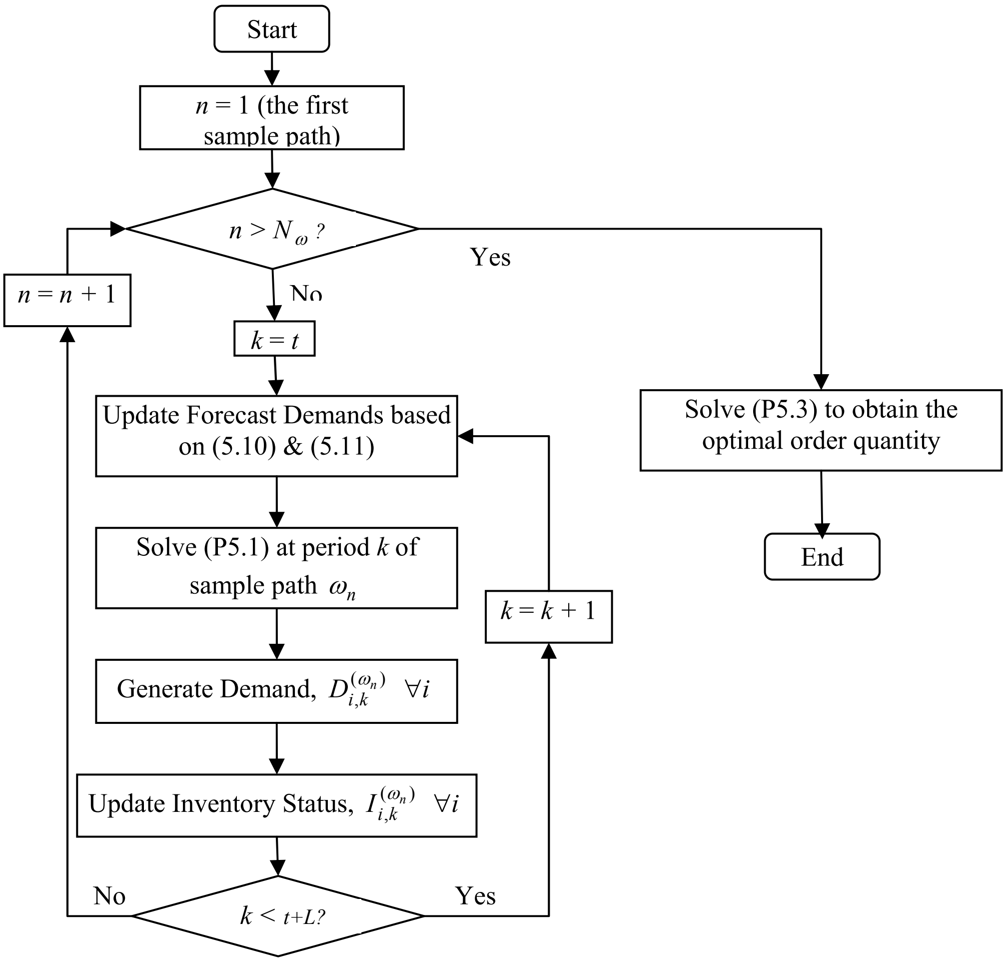


Figure 5.3: Process flow of SAA method to solve the procurement decision at period t

There is a direct interaction between the order quantity at period t and the allocation quantity at period $t+L$ of all sample paths. By adopting the SAA method, the model formulation can be derived as

(P5.3)

$$\min_{\vec{O}_t, \vec{a}_{t+L}^{(\omega_n)} \{\omega_n = \omega_1, \omega_2, \ldots \omega_N\}} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} AC_{t+L+l}^{(\omega_n)} \left(\vec{O}_t \mid \mathrm{X}_t \right)$$

$$\min_{\vec{O}_t, \vec{a}_{t+L}^{(\omega_n)} \{\omega_n = \omega_1, \omega_2, \ldots \omega_N\}} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} E\Bigg[\sum_i \left(h_i \, I_{i,t+L+l}^{+(\omega_n)} + p_i I_{i,t+L+l}^{-(\omega_n)} \right)$$

$$\sum_j \left(h_{c_j} \left(\sum_{k=t}^{t+l} O_{j,k} + \theta_{j,t+L-1}^{(\omega_n)} - \sum_i G_{ij} a_{i,t+L}^{(\omega_n)} \right) \right) | \, \mathrm{X}_t \Bigg] \tag{5.19}$$

Subject to

$$O_{j,t} + \theta_{j,t-L-1}^{(\omega_n)} - \sum_i G_{ij} a_{i,t+L}^{(\omega_n)} \geq 0 \qquad \forall j \, \& \{\omega_n = \omega_1, \omega_2, \ldots \omega_N\} \tag{5.20}$$

$$a_{i,t+L}^{(\omega_n)} \geq 0 \qquad \forall i \, \& \, \{\omega_n = \omega_1, \omega_2, \ldots \omega_N\} \tag{5.21}$$

$$O_{j,t} \quad \geq 0 \qquad \forall j \tag{5.22}$$

where $I_{i,t+L+l}^{(\omega_n)} = I_{i,t+L+l-1}^{(\omega_n)} + a_{i,t+L}^{(\omega_n)} - D_{i,t+L+l}^{(\omega_n)}$ $\forall i, \omega_n$. $\omega_n \in \Omega \;\forall n$, where n = {1,2,3,….,

N_ω }, Ω is all possible sample paths and N_ω is the number of sample paths selected to calculate the sample average value.

Theorem 5.1

(P5.3) is a convex programming problem.

Proof

The second derivatives of (5.19) with respect to the decision variables are

$$\frac{\partial^2}{\partial \left(a_{i,t}^{(\omega_n)}\right)^2} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} AC_{t+L+l}^{(\omega_n)} \left(\vec{O}_t \,|\, \mathrm{X}_t \right)$$

$$= \frac{1}{N_\omega} \left(h_i + p_i \right) f_{D_i^{l+1(\omega_n)}} (I_{i,t+L-1}^{(\omega_n)} + \sum_{k=t+L-l}^{t+L} a_{i,k}^{(\omega_n)}) \geq 0 \qquad \forall i \, \& \, \{\omega_n = \omega_1, \omega_2, \ldots \omega_N\} \tag{5.23}$$

where $f_{D_i^{l+1(\omega_n)}}(\bullet)$ is the p.d.f of $\sum_{k=t+L}^{t+L+l} D_{i,k}^{(\omega_n)}$

$$\frac{\partial^2}{\partial a_{i,t}^{(\omega_n)} a_{b,t}^{(\omega_n)}} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} AC_{t+L+l}^{(\omega_n)} \left(\vec{O}_t \,|\, \mathrm{X}_t \right) = 0 \qquad \forall i \neq b \tag{5.24}$$

$$\frac{\partial^2}{\partial a_{i,t}^{(\omega_n)} a_{i,t}^{(\omega_m)}} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} AC_{t+L+l}^{(\omega_n)}\left(\vec{O}_t \mid \mathrm{X}_t\right) = 0 \qquad \forall n \neq m \tag{5.25}$$

$$\frac{\partial^2}{\partial a_{i,t}^{(\omega_n)} a_{b,t}^{(\omega_m)}} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} AC_{t+L+l}^{(\omega_n)}\left(\vec{O}_t \mid \mathrm{X}_t\right) = 0 \qquad \forall i \neq b, n \neq m \tag{5.26}$$

$$\frac{\partial^2}{\partial a_{i,t}^{(\omega_n)} O_{j,t}^{(\omega_m)}} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} AC_{t+L+l}^{(\omega_n)}\left(\vec{O}_t \mid \mathrm{X}_t\right) = 0 \qquad \forall i, j, n, m \tag{5.27}$$

$$\frac{\partial^2}{\partial O_{j,t}^2} \frac{1}{N_\omega} \sum_{n=1}^{N_\omega} AC_{t+L+l}^{(\omega_n)}\left(\vec{O}_t \mid \mathrm{X}_t\right) = 0 \qquad \forall i \neq b, n \neq m \tag{5.28}$$

A positive semi-definite Hessian matrix is obtained. As all constraints are linear, this is a convex programming problem. Solving KKT conditions of (P5.3) yields global optimal results. □

We call the procurement policy in (P5.3) the Dynamic Level with Sharing. To quantify the benefits of component-sharing, we compared this policy with another two policies, i.e. Constant Level with Sharing and Dynamic Level without Sharing. Constant Level with Sharing has a constant order-up-to level and pools the demands of common components when determining the order-up-to level. Dynamic Level without Sharing solves the product demands independent of each other even though they share common components and allow the order-up-to level to be dynamic over periods of time.

5.4.2 Constant Level with Sharing

Constant order-up-to level means $\vec{Y}_t = \vec{Y}$ $\forall t$. The order quantity of a component is the sum of the demands that consume that component at the last period. For example, the order quantity at period t, $O_{j,t} = \sum_i G_{ij} d_{i,t-1}$ $\forall t$, where $d_{i,t-1}$ is the demand of product i

at period t-1.

For the Constant Level with Sharing policy, the problem of finding the optimal $\vec{Y}^*$ that minimizes the average total cost as given in (5.2) for the myopic component allocation policy can be formulated as

$$\text{(P5.4)} \quad \min_{\vec{Y}}\left[\frac{1}{T}\sum_{t=1}^{T} AC_{t+L+l}(\vec{Y})\right] = \min_{\vec{Y}}\left[\frac{1}{T}\sum_{t=1}^{T}\left[AC_{t+L+l}\left(\vec{a}_{t+L} \mid \mathrm{X}_{t+L}, \vec{Y}\right)\right]\right] \qquad (5.29)$$

subject to the myopic component allocation policy.

The optimal constant order-up-to levels of components can be determined by employing the following methodology. Firstly, determine the initial $\vec{Y}$. For a given $\vec{Y}$, simulation is run to estimate the average total cost as given in (5.2) and the IPA method is used to derive the gradient estimation with respect to the $\vec{Y}$. Based on the gradient information, the $\vec{Y}$ is re-adjusted, simulation is re-run and 'new' gradient estimation based on the adjusted $\vec{Y}$ is performed. This searching procedure is stopped when the stopping criteria are met. Please refer to Section 4.1.2 for more elaboration on the methodology in determining the optimum $\vec{Y}$.

5.4.3 Dynamic Level without Sharing

This policy does not consider component-sharing when making the procurement decision, but allows the order-up-to level to be dynamic. The optimal order quantity of product i at period t, $o_{i,t}$, is solved independently with respect to each product, and then the optimal order quantity of component j is the sum of the order quantity of products that need component *j*.

$$O_{j,t} = \sum_i G_{ij} o_{i,t} \qquad \forall j,\ t \qquad (5.30)$$

where $o_{i,t}$ represents the order quantity of a set of components that are used to assemble product *i*.

Similarly to the method used to solve the procurement decision for Dynamic Level with Sharing (P5.3), we employ the Monte Carlo simulation and SAA to locate the optimal order quantity of product *i*. For N_ω randomly generated sample paths, we solve the optimal order quantity of product *i* at period *t* and the allocation quantity of product *i* at period *t+L* for all generated sample paths in a single model which is defined as

(P5.5)

$$\min_{o_{i,t}, a_{i,t+L}^{(\omega_n)}\{\omega_n=\omega_1,\omega_2,\dots\omega_N\}} \frac{1}{N_\omega}\sum_{n=1}^{N_\omega} C_{t+L+l}\left(o_{i,t} \mid \mathrm{X}_t\right)$$

$$\min_{o_{i,t}, a_{i,t+L}^{(\omega_n)}\{\omega_n=\omega_1,\omega_2,\dots\omega_N\}} E\left[\left(h_i I_{i,t+L+l}^{+} + p_i I_{i,t+L+l}^{-}\right) + \sum_j \left(G_{ij} h_{c_j}\left(\sum_{k=t}^{t+l} o_{i,k} + \theta_{i,t+L-1} - a_{i,t+L}\right)\right) \mid \mathrm{X}_t\right]$$

(5.31)

subject to

$$o_{i,t} + \theta_{i,t-L-1}^{(\omega_n)} - a_{i,t+L}^{(\omega_n)} \geq 0 \ \{\omega_n = \omega_1, \omega_2, \dots\omega_N\} \qquad (5.32)$$

$$a_{i,t+L}^{(\omega_n)} \geq 0 \qquad \{\omega_n = \omega_1, \omega_2, \dots\omega_N\} \qquad (5.33)$$

$$o_{i,t} \geq 0 \qquad \forall j \qquad (5.34)$$

where $I_{i,t+L+l}^{(\omega_n)} = I_{i,t+L+l-1}^{(\omega_n)} + \sum_{k=t+L-l}^{t+L} a_{i,k}^{(\omega_n)} - \sum_{k=t+L}^{t+L+l} D_{i,k}^{(\omega_n)}$ $\forall i,\ \omega_n$. $\omega_n \in \Omega\ \forall n$, where n = {1,2,3,…., N_ω }, Ω is all possible sample paths and N_ω is the number of sample paths selected to calculate the sample average value.

Similarly to Theorem 5.1, it can be proved that (P5.5) is also a convex programming problem. Solving the KKT conditions of (P5.5) gives the optimum results.

5.5 Numerical Analysis

The benefit of component-sharing in the presence of demand correlation is evaluated by comparing the Dynamic Level with Sharing, with the Dynamic Level without Sharing. The advantage of process adjustment can be quantified by measuring the Dynamic Level with Sharing against the Constant Level with Sharing. To capture the system dynamics, simulation models are developed for the comparison. Common random numbers are used so that the comparison is based on the same set of demand pattern. This induces the same set of correlation to reduce the comparison variance. To allow the system to reach steady state performance, the data collected for the first 100 periods is discarded and is treated as warm-up period by applying Welch's procedure suggested by Law and Kelton (1991). A single set of 30,000 simulation periods is generated and divided into 30 batches, each with 1000 periods. The average total cost of each batch is employed to estimate the expected mean and variance of total cost as given in (5.2).

While our model formulation can be applied to a model with arbitrary number of components and products, we use a small example size to illustrate trend analysis for different factors. The test case consists of two common components and three products, as shown in Figure 4.3 in the previous chapter. Again, we assume that the mean demands $\mu_i = \mu$ = 50 and standard deviation $\sigma_i = \sigma$ = 10. The simulation is based on the parameter setting given in Table 5.1 by varying one parameter at each simulation batch run. The performance measures are average total cost and percentage reduction in the average total cost, taking the pure push policy as the base for comparison.

Table 5.1: Parameter setting for autoregressive demands

Parameters under Study	*Setting*
Delivery lead time: L	**1, 3, 5, 7, 10**
Assembly lead time: l	**1**
Holding cost: $h_i = h \ \forall i$	**3**
Product penalty cost: $p_i = p \ \forall i$	**10**
Component holding cost: $h_{c_j} = h_c \ \forall j$	**1**
Order of the auto-regressive process, P	**1,5,10,14**
Correlation coefficient: : $\rho_i = \rho \ \forall i$	**0, 0.1, 0.3, 0.5, 0.7, 0.9**

Truncated normal distribution is used for the demand distribution, where any negative random number generated as product demand is discarded. The maximum number of iterations is 1000. The number of sample paths selected to calculate the sample average value is 30. Assuming only one unit of component j is used, G_{ij} is 1 when product i uses component j and 0 otherwise.

The standardized format for inventory on-hand (Rogers and Tsubakitani, 1991) is applied to facilitate the total cost calculation; for example from (5.16),

$$E[I^{+}_{i,t+L+l}] = \int_{-\infty}^{I_{i,t+L-1}+\sum_{k=t+L-l}^{t+L} a_{i,k}} \left(I_{i,t+L-1} + \sum_{k=t+L-l}^{t+L} a_{i,k} - u_i \right) f_{D_i^{l+1}}(u_i)\, du_i$$

$$= \sqrt{\sigma'^2_i \, \Gamma(\vec{d}, i, l+1)} \left(Z_{i,t} + R(Z_{i,t}) \right) \tag{5.35}$$

where $\sqrt{\sigma'^2_i \, \Gamma(\vec{d}, i, l+1)}$ is the standard deviation of the lead time demand of product i

for l+1 periods. $f_{D_i^{l+1}}(\bullet)$ is the p.d.f of $\sum_{k=t+L}^{t+L+l} D_{i,k}$. $Z_{i,t}$ is the standardized inventory position at the end of period t, $Z_{i,t} = \dfrac{\left(I_{i,t+L-1} + \sum_{k=t+L-l}^{t+L} a_{i,k} - u_i(l+1) - \Lambda(d,i,l+1)\right)}{\sqrt{\sigma'^2_i \, \Gamma(\vec{d},i,l+1)}}$.

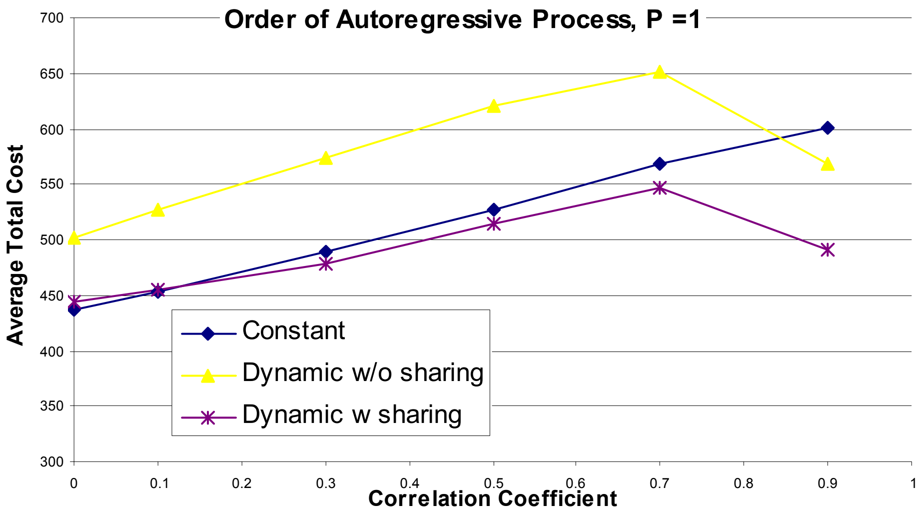
$R(Z_{i,t})$ is the right-hand unit normal loss integral, $R(Z_{i,t}) = \int_{Z_{i,t}}^{\infty} \frac{\overline{u}_i - Z_{i,t}}{\sqrt{2\pi}} e^{-\frac{\overline{u}_i^2}{2}} d\overline{u}_i$ and $\overline{u}_i = \dfrac{u_i - u_i(l+1) - \Lambda(d,i,l+1)}{\sqrt{\sigma'^2_i \, \Gamma(\vec{d},i,l+1)}}$.

Similarly, the number of back-orders at the end of the lead time is given by

$$E\left[I^-_{i,t+L+l}\right] = \int_{I_{i,t+L-1} + \sum_{k=t+L-l}^{t+L} a_{i,k}}^{\infty} \left(u_i - I_{i,t+L-1} - \sum_{k=t+L-l}^{t+L} a_{i,k}\right) f_{D_i^{l+1}}(u_i)\,du_i$$

$$= \sqrt{\sigma'^2_i \, \Gamma(\vec{d},i,l+1)}\left(R(Z_{i,t})\right) \tag{5.36}$$

Detailed derivation and the alternative form of $R(Z_{i,t})$ for numerical evaluation are explained in Appendix D.



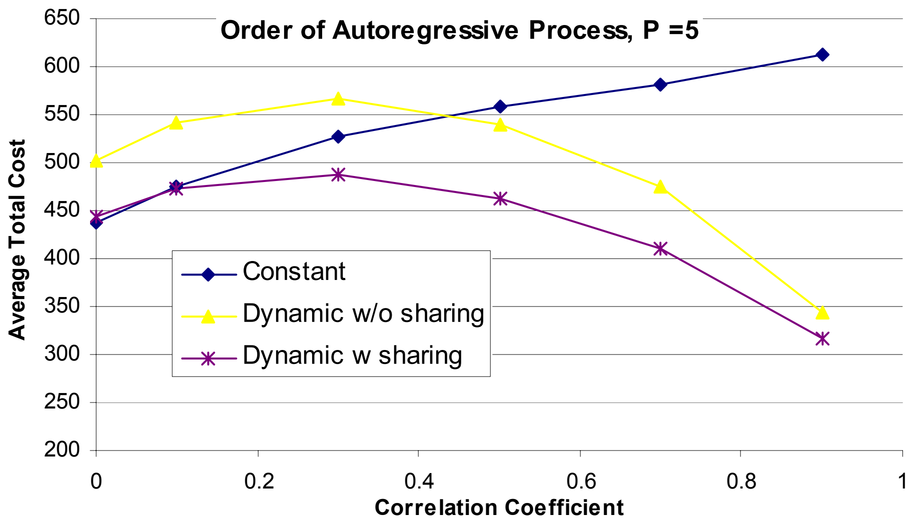
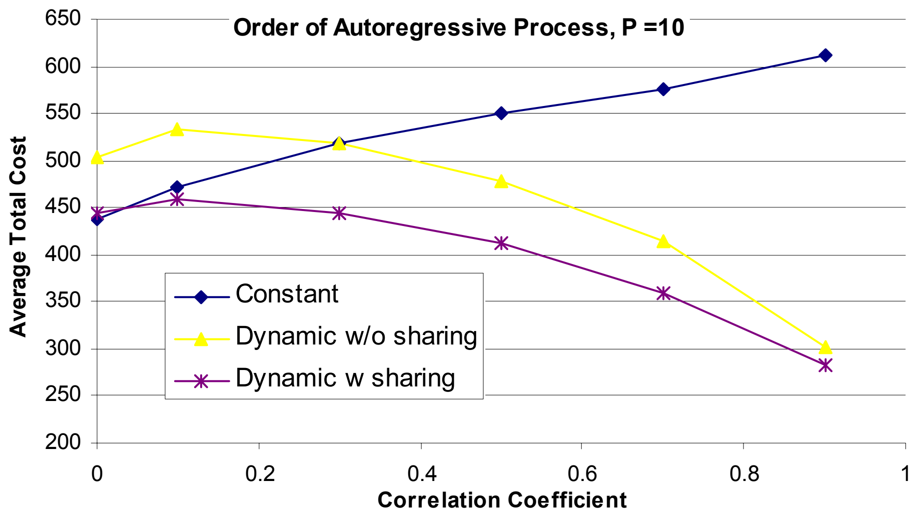


Figure 5.4: Average total cost of different policies when $L = 5$



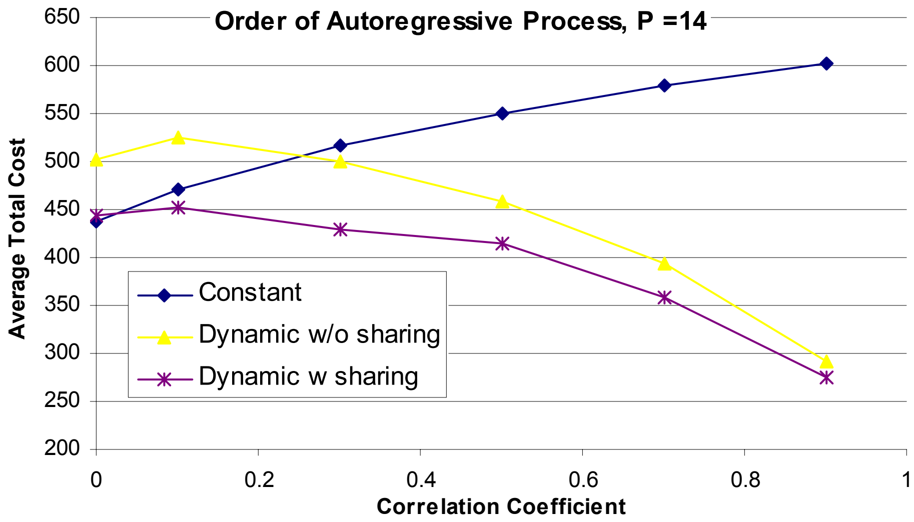


Figure 5.4: Average total cost of different policies when $L = 5$ (continued)

The performance in terms of average total cost for three procurement policies is plotted in Figure 5.4 with respect to different level of correlation coefficient and order of the auto-regressive process. Because the graph of Dynamic Level with Sharing is always below the graph of Dynamic Level without Sharing, Figure 5.4 illustrates that the Dynamic Level with Sharing always performs better than the Dynamic Level without Sharing due to the risk-pooling effect of common components. The marginal cost difference decreases as the correlation coefficient, ρ, or order of the auto-regressive process, P, increases or both. This is because, as ρ and P increase, the demands becomes more positively correlated with the previous periods. A higher than expected demand is more likely to be followed by higher than expected demand in the subsequent periods and therefore request more components for a number of periods. Similarly, a lower than expected demand is more likely to be followed by lower than expected demand in the subsequent periods. This positive correlation has negated the risk-pooling effect as the policy has less scope to share different common components by temporary 'borrowing' or 'swapping' components among products that share the same common components.

By measuring the performance of Dynamic Level with Sharing against the Constant Level with Sharing, we can quantify the effect of having dynamic order-up-to level over the constant order-up-to level. For ease of reference, we categorize this as adjustment effect subsequently. When ρ is small, Constant Level with Sharing can achieve a lower cost over the Dynamic Level with Sharing, but, the marginal increment in cost is very small and statistically the difference is insignificant (see Appendix H for details). This can be attributed to the randomness in simulation. Furthermore, the Dynamic Level with Sharing looks at the myopic cost when determining the order quantity, whereas the Constant Level with Sharing looks at the impact of the order-up-to level over the whole planning horizon. The cost-savings due to adjustment of order-up-to level increases as the correlation coefficient, ρ, increases. The graph of Constant Level with Sharing goes up, while the graph of Dynamic Level with Sharing goes down. The cost-savings or the gap between the two graphs becomes wider.

Figure 5.5 depicts the adjustment effect and component-sharing effect for $L = 5$. This infers that negative cost-savings means an increment in cost. The results show that when ρ and P are small, the component-sharing effect gives more cost-savings than the adjustment effect. However, the component-sharing effect becomes more dominant when either ρ or P is big. In addition, the advantage of the component-sharing effect abates as the order of the auto-regressive process, P, increases. On the other hand, the benefit of the adjustment effect is amplified by the order of the auto-regressive process, P, but at a slower rate. From Figure 5.5, the component-sharing effect is more dominant than the adjustment effect for a given P when ρ is small. As ρ increases, the component-sharing effect diminishes while the adjustment effect becomes more pronounce. If ρ increases beyond the interception point between the 'Adjustment' graph and the 'Sharing' graph, the adjustment effect becomes more dominant.

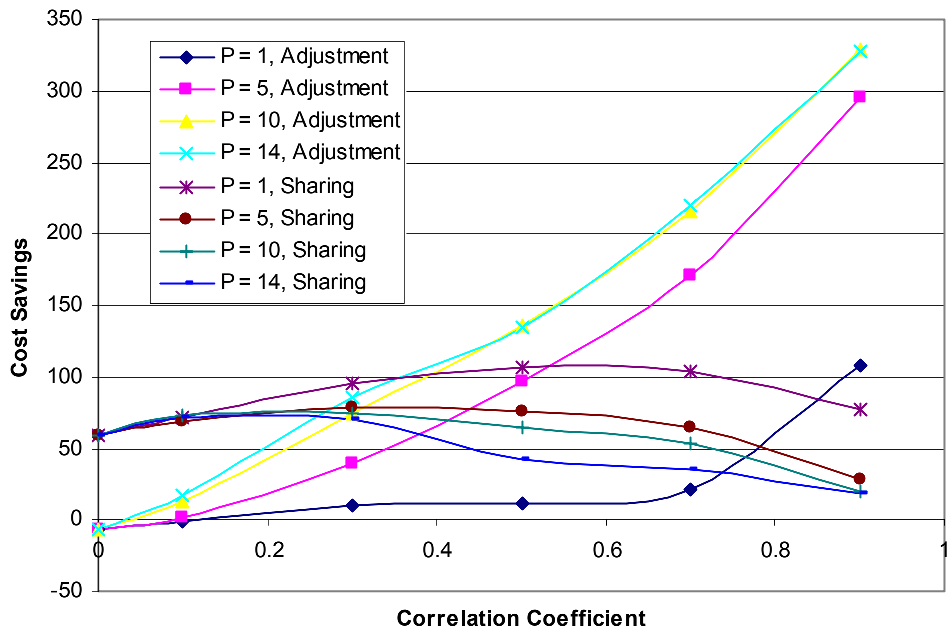


Figure 5.5: Cost-savings from adjustment and component-sharing for different P when $L = 5$.

By varying the delivery lead time, results similar to the above can be reached as shown in Figure 5.6. For Dynamic Level with Sharing, the average total cost is much lower than for Dynamic Level without Sharing. The marginal cost is magnified by the delivery lead time L. The component-sharing effect diminishes as ρ increases, but increases in L. Figure 5.6 depicts that the correlation coefficient will have higher impact on the adjustment effect than the component-sharing as L increases. Overall, Dynamic Level with Sharing should be recommended to be employed because it has the lowest average total cost.

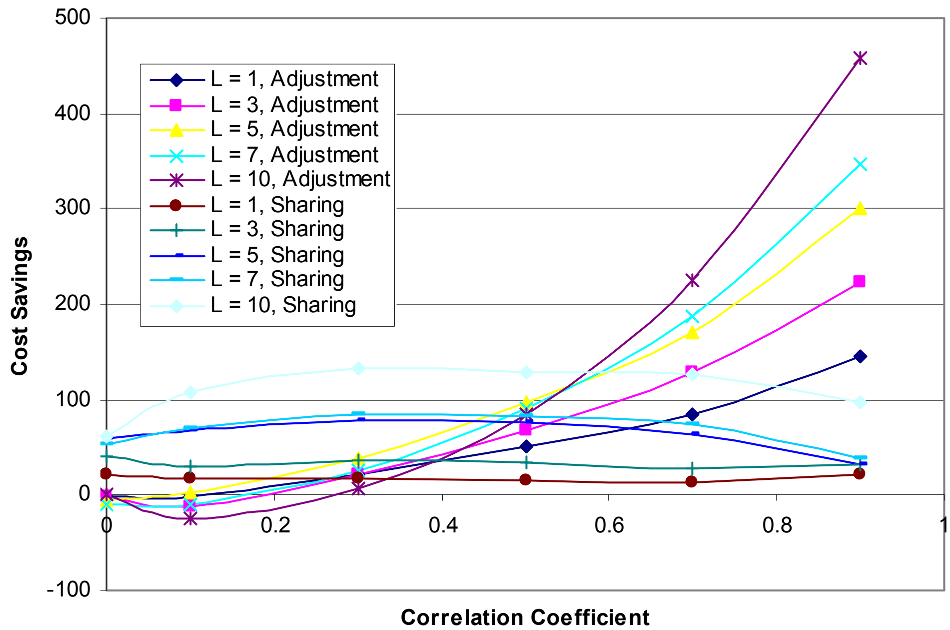


Figure 5.6: Cost-savings from correlation effect and component-sharing for different L when $P = 5$.

5.6 Summary

Three procurement policies have been studied to identify the component-sharing effect and adjustment effect. The demands are modeled as an auto-regressive process. Comparatively, the adjustment effect shows a more substantial saving in the average total cost for positively correlated demands than the component-sharing effect. This is because the risk-pooling effect dwindles as the demands become more positively correlated while the adjustment effect increases in ρ and P. Even though the cost-savings of component-sharing is relatively smaller than the adjustment effect when ρ and P are large, it is recommended that the myopic allocation policy be used in conjunction with dynamic order-up-to level to fully realize the potential of the component-sharing effect.

CHAPTER 6 CONCLUSIONS

6.0 Concluding Remarks

Assemble-To-Stock (ATS) and the use of common components, where the components are common to a set of distinct products, are the norm for today's manufacturing environment. In ATS systems, production begins before demand is precisely known. ATS companies generally produce in batches and carry finished goods' inventories for most items. The advantage is that customer delivery times are minimized. There is, however, an expense of inventory holding costs: in ATS, the customers are not willing to wait for their requirements; the manufacturer has to be ready to sell 'off the shelf' when a customer demands a product or risk losing the sale. Common components can bring a lot of quantifiable positive economic impacts. Among the benefits are ease of inventory management, reduced new product development cost, economies of scale in purchasing material, and improvement in forecast accuracy due to demand aggregation.

6.1 Main Findings and Contributions

To date, there are limited publications addressing ATS systems with multiple common components. Our research has managed to fill this gap and quantify the benefits of component-sharing in an ATS system with

- Long delivery lead time
- Multiple periods
- Multiple common components

We have modeled a two-echelon supply chain model with a long procurement lead times for components. These components are subsequently used in the assembly of several products in an ATS environment. A periodic review inventory policy is assumed. There are two decisions to make at every period. Firstly, the procurement decision as to how many components to procure, and secondly, the allocation decision as to how many components to release into the assembly line to make a product. The research is carried out in a phased approach.

In the first phase (Chapter 3), we have shown the tradeoff between the risk-pooling effect and matching problem. These effects are analyzed by comparing an equal fractile allocation policy which exploits component-sharing against a policy that does not allow component-sharing. The probabilistic analysis has proved that the equal fractile allocation policy becomes dominant when all components can be shared by at least two products with a high service standard. We have shown that, under certain conditions in Theorem 3.2, the expected cost model for an equal fractile allocation policy can be reduced to a newsvendor problem

In the second phase (Chapter 4), we have proposed a component allocation policy that minimizes the conditional expectation of the total cost for a corresponding future period based on the system state and the given order-up-to levels. The proposed allocation policy allows component-sharing. The allocation model is a convex programming problem at any given period. We have introduced a simulation-based optimization method to identify the optimal order-up-to levels of components. The numerical results reveal that the proposed policy consistently yields a lower total cost than the two-echelon policy because of component-sharing. Except in some cases when the incremental holding cost is high, the saving through component-sharing is quite substantial and is almost on par with the saving through the echelon effect.

In the third phase (Chapter 5), we have introduced auto-correlation into the demand process and proposed a dynamic order-up-to level policy that considered component-sharing when making the procurement decision. A comparative study among three distinct procurement policies is conducted to identify the component-sharing effect and the adjustment effect. The saving resulted from component-sharing is higher than the adjustment effect when the correlation coefficient and order of auto-regressive are small. However, the adjustment effect becomes more influential on the total cost than the component-sharing effect when the correlation coefficient is closer to 1 and the order of auto-regressive is high.

In summary, we have proposed a component allocation policy and procurement policy that consider component-sharing. We have also quantified the benefits of component-sharing in the allocation policy and procurement policy by comparing the policy with another policy that does not consider component-sharing. The proposed policies are easy to implement. Constant order-up-to level with sharing is recommended if demands are not auto-correlated, whereas dynamic order-up-to level with sharing is recommended in the presence of auto-correlated demands. The contributions at different phases are summarized in the below diagram.

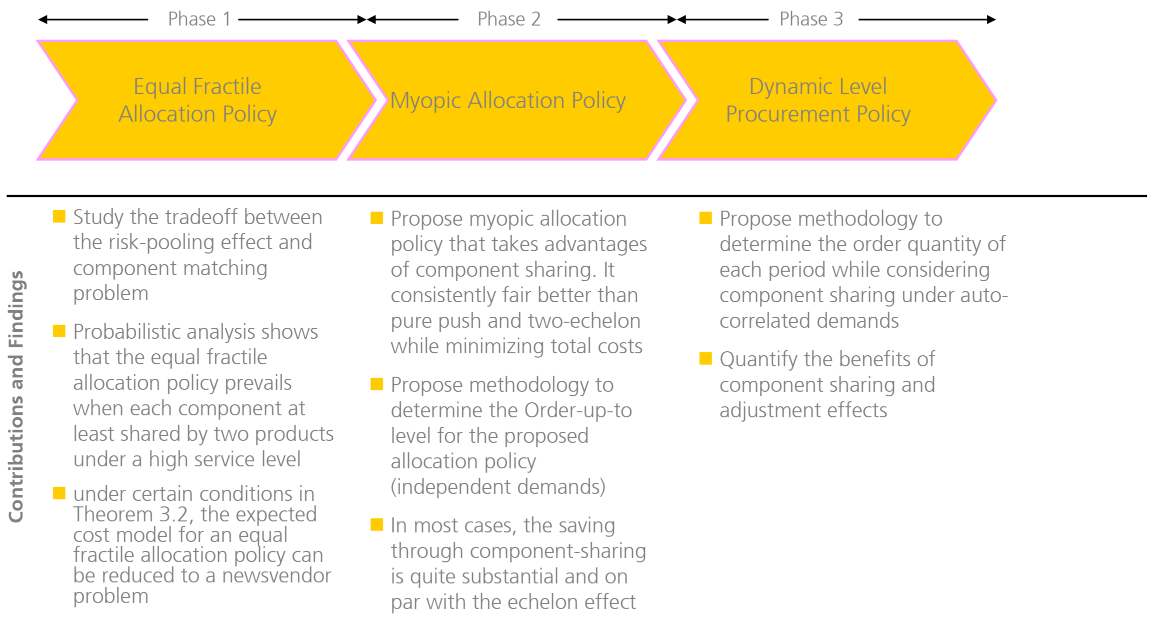


Figure 6.1: Summary of the main contributions and findings.

6.2 Suggestions for Further Research

This research has proposed a component allocation policy that minimizes the total cost of a future period when those components allocated at that period become available to fill the actual demands. An extension to this research can focus on the modelling of another component allocation policy that minimizes the number of periods of total cost when determining the allocation decision and comparing its performance against the myopic allocation policy. For instance, the "new" component allocation policy will attempt to reduce the total cost of period $t+l$ and subsequent periods when determining the allocation decision. Dynamic programming may be applied to solve the allocation model that applies discounted rates for the total cost of future periods. Simple test case scenarios may be adopted for comparison against findings concluded in this research.

Similar thought can be adopted in the Dynamic Level with Sharing, the proposed model looks at the total cost of a future period. A possible extension is to consider multi-period cost simultaneously.

As discussed in Chapter 3, we have developed a lower bound for the equal fractile allocation policy in which a couple of assumptions is imposed in order to facilitate the mathematical tractability of the model. However, this has reduced the practicality of the policy. Furthermore, we have not addressed the issue of bounds in the measurement of the performance of our proposed myopic allocation policy and the Dynamic Level with Sharing policy with respect to the optimal solution in our research. The above stand is taken as we have considered that the optimal policies will not be tractable and the model will become too complex to resolve in our context. Thus, more time and effort have to be invested and this is a potential challenge for future research direction.

In addition, this research can be further extended to analyze the effect of assembly capacity. We have earlier assumed that the assembly capacity is unlimited and it may not be the case in reality. By introducing the capacity constraint at individual period, finished stock may need to be built up in advance of peak season and reduce risk pooling opportunity.

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Appendix A: Clark's Approximation Method

Clark (1961) has given exact expressions for the first four moments of the maximum of a pair of jointly normal variants and the correlation coefficient between the maximum of the pair of two normal random variables.

Before proceeding, let us discuss the formulation of Clark Algorithm. Let $\xi \sim N(\mu_1, \sigma_1^2)$, $\eta \sim N(\mu_2, \sigma_2^2)$ and $\tau \sim N(E(\tau), V(\tau))$. If r denotes the coefficient of linear correlation, $r(\xi, \eta) = \rho$, $r(\xi, \tau) = \rho_1$ and $r(\eta, \tau) = \rho_2$. Define notation $\varphi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ and $\Phi(x) = \int_{-\infty}^{x}\varphi(t)dt$. This derivation does not applies when $\rho = 1$.

Let v_i be the i-th moment (about zero) of the random variable $\max(\xi, \eta)$. The following notation is defined

$$a^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho \tag{A.1}$$

This expression is positive when $\rho \neq 1$. Introducing notation

$$\alpha = (\mu_1 - \mu_2)/a \tag{A.2}$$

Clark (1961) has proved that

$$v_1 = \mu_1\Phi(\alpha) + \mu_2\Phi(-\alpha) + a\varphi(\alpha) \tag{A.3}$$

$$v_2 = \left(\mu_1^2 + \sigma_1^2\right)\Phi(\alpha) + \left(\mu_2^2 + \sigma_2^2\right)\Phi(-\alpha) + \left(\mu_1 + \mu_2\right)a\varphi(\alpha) \tag{A.4}$$

$$\begin{aligned} v_3 = {} & \left(\mu_1^3 + 3\mu_1\sigma_1^2\right)\Phi(\alpha) + \left(\mu_2^3 + 3\mu_2\sigma_2^2\right)\Phi(-\alpha) + [\left(\mu_1^2 + \mu_1\mu_2 + \mu_2^2\right)a \\ & + \left(2\sigma_1^4 + \sigma_1^2\sigma_2^2 + 2\sigma_2^4 - 2\sigma_1^3\sigma_2\rho - 2\sigma_1\sigma_2^3\rho - \sigma_1^2\sigma_2^2\rho\right)a^{-1}]\varphi(\alpha) \end{aligned} \tag{A.5}$$

$$\begin{aligned} v_4 = {} & \left(\mu_1^4 + 6\mu_1^2\sigma_1^2 + 3\sigma_1^4\right)\Phi(\alpha) + \left(\mu_2^4 + 6\mu_2^2\sigma_2^2 + 3\sigma_2^4\right)\Phi(-\alpha) \\ & + \{\left(\mu_1^3 + \mu_1^2\mu_2 + \mu_1\mu_2^2 + \mu_2^3\right)a - 3\alpha\left(\sigma_1^4 - \sigma_2^4\right) + 4\mu_1\sigma_1^3[3\left(\sigma_1 - \sigma_2\rho\right)/a \\ & - \left(\sigma_1 - \sigma_2\rho\right)^3/a^3] + 4\mu_2\sigma_2^3[3\left(\sigma_2 - \sigma_1\rho\right)/a - \left(\sigma_2 - \sigma_1\rho\right)^3/a^3]\}\varphi(\alpha) \end{aligned} \tag{A.6}$$

$$r[\tau, \max(\xi, \eta)] = [\sigma_1\rho_1\Phi(\alpha) + [\sigma_2\rho_2\Phi(-\alpha)]/(v_2 - v_1^2)^{1/2} \tag{A.7}$$

The formula for the v_i permits calculations related to the greater two normal variables. Equation (A.7) is used in estimating moments of the greater of more than two normally distributed variables.

For a two-component and three-product case,

$u_1 = \xi_1 / (\sigma_1 + \sigma_2)$ and $u_2 = \xi_2 / (\sigma_2 + \sigma_3)\, u_2$

where $\xi_1 \sim N[0, L(\sigma_1^2 + \sigma_2^2)]$ and $\xi_2 \sim N[0, L(\sigma_2^2 + \sigma_3^2)]$.

Covariance (u_1, u_2)

$= \mathrm{E}[(u_1 - \mathrm{E}[u_1])(u_2 - \mathrm{E}[u_2])] = \mathrm{E}[u_1 u_2] - \mathrm{E}[u_1]\,\mathrm{E}[u_2]$

$= \mathrm{E}[u_1 u_2]$ (as $\mathrm{E}[u_1] = \mathrm{E}[u_2] = 0$)

$$= \frac{E[\xi_1 \xi_2]}{(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)} \qquad \text{(A.8)}$$

Let q1 $\sim N[0, L(\sigma_1^2)]$, q2 $\sim N[0, L(\sigma_2^2)]$ and q3 $\sim N[0, L(\sigma_3^2)]$

$E[\xi_1 \xi_2]$ = Covariance $[\xi_1 \xi_2]$ = E [(q1 + q2)(q2 + q3)] – E[q1 + q2] E[q2 + q3]

= E[(q1 q2 + q1 q3 +q2^2 + q2 q3)] – 0

= 0 + 0 + (0 + $L\sigma_2^2$) + 0

= $L\sigma_2^2$

Equation (A.8) will become,

$$\text{Covariance } (u_1, u_2) = \frac{L\sigma_2^2}{(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)}$$

The coefficient of correlation is

$$\rho = \frac{Cov(u_1, u_2)}{\sqrt{Var(u_1)Var(u_2)}}$$

$$r(u_1, u_2) = \rho = \frac{L\sigma_2^2}{(\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)} \frac{(\sigma_1 + \sigma_2)}{\sqrt{L(\sigma_1^2 + \sigma_2^2)}} \qquad \frac{(\sigma_2 + \sigma_3)}{\sqrt{L(\sigma_2^2 + \sigma_3^2)}}$$

$$= \frac{\sigma_2^2}{\sqrt{(\sigma_1^2 + \sigma_2^2)(\sigma_2^2 + \sigma_3^2)}} \qquad \text{(A.9)}$$

From Clark Algorithm, Equation (A.1),

$$a^2 \quad = \frac{L\left(\sigma_1^2+\sigma_2^2\right)}{\left(\sigma_1+\sigma_2\right)^2} + \frac{L\left(\sigma_2^2+\sigma_3^2\right)}{\left(\sigma_2+\sigma_3\right)^2} - 2\frac{L\sigma_2^2}{(\sigma_1+\sigma_2)(\sigma_2+\sigma_3)} \qquad \text{(A.10)}$$

From Equation (A.2),

$$\alpha \quad = \frac{E[u_2]-E[u_1]}{a} = 0$$

From Equation (A.3), the first moment of the random variable η is

$$E[\eta] \quad = \nu_1 \quad = \frac{a}{\sqrt{2\pi}}$$

$$= \sqrt{\frac{L}{2\pi}\left(\frac{(\sigma_1^2+\sigma_2^2)}{\left(\sigma_1+\sigma_2\right)^2}+\frac{(\sigma_2^2+\sigma_3^2)}{\left(\sigma_2+\sigma_3\right)^2}-\frac{2\sigma_2^2}{\left(\sigma_1+\sigma_2\right)\left(\sigma_2+\sigma_3\right)}\right)} \qquad \text{(A.11)}$$

From Equation (A.4), the second moment of the random variable η is

$$\nu_2 = \frac{L\left(\sigma_1^2+\sigma_2^2\right)}{2\left(\sigma_1+\sigma_2\right)^2} + \frac{L\left(\sigma_2^2+\sigma_3^2\right)}{2\left(\sigma_2+\sigma_3\right)^2} \qquad \text{(A.12)}$$

As $\Phi(0) = 0.5$

The variance of the random variable η is

$$Var(\eta) \quad = \nu_2 - \nu_1^2$$

$$= \frac{L\left(\sigma_1^2+\sigma_2^2\right)}{2\left(\sigma_1+\sigma_2\right)^2} + \frac{L\left(\sigma_2^2+\sigma_3^2\right)}{2\left(\sigma_2+\sigma_3\right)^2} - \frac{L}{2\pi}\left(\frac{(\sigma_1^2+\sigma_2^2)}{\left(\sigma_1+\sigma_2\right)^2}+\frac{(\sigma_2^2+\sigma_3^2)}{\left(\sigma_2+\sigma_3\right)^2}-\frac{2\sigma_2^2}{\left(\sigma_1+\sigma_2\right)\left(\sigma_2+\sigma_3\right)}\right)$$

$$= L\left(\frac{(\sigma_1^2+\sigma_2^2)}{2\left(\sigma_1+\sigma_2\right)^2}+\frac{(\sigma_2^2+\sigma_3^2)}{2\left(\sigma_2+\sigma_3\right)^2}\right)\left(1-\frac{1}{\pi}\right)+\frac{L\sigma_2^2}{\pi\left(\sigma_1+\sigma_2\right)\left(\sigma_2+\sigma_3\right)} \qquad \text{(A.13)}$$

We can determine exactly the expected value and the standard deviation of η given in (3.28) by Equation (A.12) and (A.13), respectively.

From (3.15), the normalized net inventory of product is

$$X_i / \sigma_i = K_1 \sqrt{L + l + 1} - \eta - \vartheta$$

where $\vartheta \sim N(0, l+1)$.

From (3.16), the service level of products for the equal fractile allocation policy is

$$\Pr(X_i / \sigma_i \geq 0 \mid K_1) = \Pr\left(\eta + \vartheta \leq K_1 \sqrt{L + l + 1}\right)$$

If we approximate the random variable η to be normally distributed, then the distribution of the normalized net inventory of products given by (3.15) is normally distributed.

$$\Pr\left(\eta + \vartheta \leq K_1 \sqrt{L + l + 1}\right) = \Phi\left(\frac{K_1 \sqrt{L + l + 1} - E[\eta]}{\sqrt{Var(\eta) + l + 1}}\right)$$

Hence, the relationship between K_2 and K_1 can be found in (A.15).

$$\Phi\left(\frac{K_1 \sqrt{L + l + 1} - E[\eta]}{\sqrt{Var(\eta) + l + 1}}\right) = \Phi(K_2) \qquad \text{(A.14)}$$

or

$$\frac{K_1 \sqrt{L + l + 1} - E[\eta]}{\sqrt{Var(\eta) + l + 1}} = K_2 \qquad \text{(A.15)}$$

The comparison between the two policies is made in terms of the difference in the normalized safety stock as given in (3.20). $\Delta = K_2 - K_1$ can be estimated by Equation (A.16).

$$\Delta = \frac{K_1 \left[\sqrt{L + l + 1} - \sqrt{(Var(\eta) + l + 1)}\right] - E[\eta]}{\sqrt{(Var(\eta) + l + 1)}} \qquad \text{(A.16)}$$

For a higher dimension of components and products, Clark (1961) has provided an approximation to estimate the parameters of the maximum value of a general number of normal random variables. Again, we can obtain E[η] and $Var(\eta)$ by approximating η to follow a normal distribution and use the approximation to make the comparison.

Appendix B: Proof of Optimality Condition for Equation (3.38)

Consider the minimization problem, since $X_i = -X'$, Equation (3.38) can be rewritten as

$$\min_{Q_1,\ldots,Q_J} \text{TC} = \min_{Q_1,\ldots,Q_J} -Ih' \int_{-\infty}^{0} x f_{X'}(x)dx + I(p + mh_c) \int_{0}^{\infty} x f_{X'}(x)dx + h_c \sum_j Q_j \qquad \text{(B.1)}$$

The probability density function of X' is $f_{X'}(x) = \int_{-\infty}^{+\infty} f_{\eta'}(x-z) f_{\vartheta'}(z)dz$

where

$$f_{\eta'}(v) = \frac{\partial F_{\eta'}(v)}{\partial v}$$

$$= \frac{\partial \int_{-\infty}^{v} \ldots \int_{-\infty}^{v} \frac{1}{(2\pi)^{n/2} |\Sigma_U|^{1/2}} \exp\left(-\frac{1}{2} (\vec{U}_j - \vec{M})^T \Sigma_U^{-1} (\vec{U}_j - \vec{M}) \right) du_1 \ldots\ldots du_J}{\partial v}$$

By applying Multiplication rule (Abramowitz and Stegun, 1972; Silver et al., 1998),

$$= \frac{\int_{-\infty}^{v} \ldots \int_{-\infty}^{v} \partial \int_{-\infty}^{v} \frac{1}{(2\pi)^{n/2} |\Sigma_U|^{1/2}} \exp\left(-\frac{1}{2} (\vec{U}_j - \vec{M})^T \Sigma_U^{-1} (\vec{U}_j - \vec{M}) \right) du_1 du_2 \ldots\ldots du_J}{\partial v} +$$

$$\frac{\int_{-\infty}^{v} \ldots \int_{-\infty}^{v} \partial \int_{-\infty}^{v} \frac{1}{(2\pi)^{n/2} |\Sigma_U|^{1/2}} \exp\left(-\frac{1}{2} (\vec{U}_j - \vec{M})^T \Sigma_U^{-1} (\vec{U}_j - \vec{M}) \right) du_2 du_1 du_3 \ldots\ldots du_J}{\partial v} +$$

$$\frac{\int_{-\infty}^{v} \ldots \int_{-\infty}^{v} \partial \int_{-\infty}^{v} \frac{1}{(2\pi)^{n/2} |\Sigma_U|^{1/2}} \exp\left(-\frac{1}{2} (\vec{U}_j - \vec{M})^T \Sigma_U^{-1} (\vec{U}_j - \vec{M}) \right) du_j du_1 \ldots du_{j-1} du_{j+1} \ldots\ldots du_J}{\partial v} +$$

$$\frac{\int_{-\infty}^{v} \ldots \int_{-\infty}^{v} \partial \int_{-\infty}^{v} \frac{1}{(2\pi)^{n/2} |\Sigma_U|^{1/2}} \exp\left(-\frac{1}{2} (\vec{U}_j - \vec{M})^T \Sigma_U^{-1} (\vec{U}_j - \vec{M}) \right) du_J du_1 \ldots\ldots du_{J-1}}{\partial v}$$

(B.2)

where $\vec{U}'_j =$ $(u_1, \ldots, u_j = v, \ldots, u_J)$ and $\vec{M} = (-Q_1/m, \ldots, -Q_J/m)$.

By applying Leibnitz's rule (Abramowitz and Stegun, 1972), we can derive the following equation

$$\frac{\int_{-\infty}^{v}\cdots\int_{-\infty}^{v}\partial\int_{-\infty}^{v}\frac{1}{(2\pi)^{n/2}|\Sigma_U|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}_j-\vec{M})\right)du_1 du_2\ldots\ldots du_J}{\partial v}+$$

$$=\left[\int_{-\infty}^{v}\cdots\cdots\int_{-\infty}^{v}\frac{\partial\frac{1}{(2\pi)^{n/2}|\Sigma_U|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}_j-\vec{M})\right)du_1\ldots\ldots du_J}{\partial v}+\right.$$

$$\int_{-\infty}^{v}\cdots\int_{-\infty}^{v}\frac{1}{(2\pi)^{n/2}|\Sigma_U|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}'_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}'_j-\vec{M})\right)du_2\ldots\ldots du_J\frac{\partial v}{\partial v}\ -$$

$$\left.\int_{-\infty}^{v}\cdots\int_{-\infty}^{v}\frac{1}{(2\pi)^{n/2}|\Sigma_U|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}'_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}'_j-\vec{M})\right)du_2\ldots\ldots du_J\frac{\partial-\infty}{\partial v},\ j=1,2\ldots.J\right]$$

$$=\int_{-\infty}^{v}\cdots\int_{-\infty}^{v}\frac{1}{(2\pi)^{n/2}|\Sigma_U|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}'_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}'_j-\vec{M})\right)du_2\ldots\ldots du_J$$

(B.3)

By applying Leibnitz's rule in (B.2), we have

$$=\sum_{j=1}^{J}\int_{-\infty}^{v}\cdots\int_{-\infty}^{v}\frac{1}{(2\pi)^{n/2}|\Sigma_U|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}'_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}'_j-\vec{M})\right)du_1\ldots du_{-j}\ldots du_J$$

(B.4)

where du_{-j} represents without the integral function of $\int_{-\infty}^{v}du_j$.

From (B.1) and (B.4), the partial differential of TC with respect to Q_r, $r = 1,\ldots,J$

$$\frac{\partial TC}{\partial Q_r} = -Ih'\int_{-\infty}^{0} x\frac{\partial f_{X'}(x)}{\partial Q_r}dx + I(p+mh_c)\int_{0}^{\infty} x\frac{\partial f_{X'}(x)}{\partial Q_r}dx + h_c \tag{B.5}$$

where

$$\frac{\partial f_{X'}(x)}{\partial Q_r} =$$

$$\sum_{j=1}^{J}\int_{-\infty}^{+\infty}\int_{-\infty}^{x-z}\int_{-\infty}^{x-z}\frac{\left(\vec{U}'_j-\vec{M}\right)^T\vec{a}_r}{m(2\pi)^{n/2}\left|\Sigma_U\right|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}'_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}'_j-\vec{M})\right)f_{\vartheta}(z)du_1\ldots du_{-j}\ldots du_J dz \tag{B.6}$$

where $\vec{U}'_j = (u_1, \ldots, u_j = x-z, \ldots, u_J)$, $\vec{M} = (-Q_1/m, \ldots, -Q_J/m)$ and $\vec{a}_r$ is the rth column of Σ_U^{-1}.

Expressing (B.6) as

$$\frac{\partial f_{X'}(x)}{\partial Q_r} = G_1(x, Q_j; j=1,\ldots,J) - \vec{M}^T\vec{a}_r\, G_2(x, Q_j; j=1,\ldots,J) \tag{B.7}$$

where

$G_1(x, Q_j; j=1,\ldots,J) =$

$$\sum_{j=1}^{J}\int_{-\infty}^{+\infty}\int_{-\infty}^{x-z}\int_{-\infty}^{x-z}\frac{\vec{U}'^{\,T}_j\vec{a}_r}{m(2\pi)^{n/2}\left|\Sigma_U\right|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}'_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}'_j-\vec{M})\right)f_{\vartheta}(z)du_1\ldots du_{-j}\ldots du\ dz$$

and

$G_2(x, Q_j; j=1,\ldots,J) =$

$$\sum_{j=1}^{J}\int_{-\infty}^{+\infty}\int_{-\infty}^{x-z}\int_{-\infty}^{x-z}\frac{1}{m(2\pi)^{n/2}\left|\Sigma_U\right|^{1/2}}\exp\left(-\frac{1}{2}(\vec{U}'_j-\vec{M})^T\Sigma_U^{-1}(\vec{U}'_j-\vec{M})\right)f_{\vartheta}(z)du_1\ldots du_{-j}\ldots du\ dz$$

Then, Equation (B.5) can be written as

$$\frac{\partial TC}{\partial Q_r} = -Ih'\int_{-\infty}^{0} xG_1(x,Q_j; j=1,\ldots,J)dx + I(p+mh_c)\int_{0}^{\infty} xG_1(x,Q_j; j=1,\ldots,J)dx + h_c$$

$$+\vec{M}^T\vec{a}_r\left(Ih'\int_{-\infty}^{0}xG_2(x,Q_j;j=1,...,J)dx-I(p+mh_c)\int_{0}^{\infty}xG_2(x,Q_j;j=1,...,J)dx\right)$$

The stationary points occur at

$$\vec{M}^T\vec{a}_r=\frac{Ih'\int_{-\infty}^{0}xG_1(x,Q_j;j=1,...,J)dx-I(p+mh_c)\int_{0}^{\infty}xG_1(x,Q_j;j=1,...,J)dx-h_c}{Ih'\int_{-\infty}^{0}xG_2(x,Q_j;j=1,...,J)dx-I(p+mh_c)\int_{0}^{\infty}xG_2(x,Q_j;j=1,...,J)dx}$$

$$r=1,\dots,J \qquad \text{(B.8)}$$

Let RHS of (B.8) denote as $C(Q_j, j = 1,\dots,J)$.

$$C(Q_j,j=1..J)=\frac{Ih'\int_{-\infty}^{0}xG_1(x,Q_j;j=1,...,J)dx-I(p+mh_c)\int_{0}^{\infty}xG_1(x,Q_j;j=1,...,J)dx-h_c}{Ih'\int_{-\infty}^{0}xG_2(x,Q_j;j=1,...,J)dx-I(p+mh_c)\int_{0}^{\infty}xG_2(x,Q_j;j=1,...,J)dx}$$

Equation (B.8) can be obtained by solving J simultaneous equations

$$\vec{M}^T\Sigma_U^{-1}=C(Q_j,j=1,\dots,J)\vec{1}^T$$

$$\vec{M}^T=C(Q_j,j=1,\dots,J)\vec{1}^T\Sigma_U$$

or $\quad -Q_r/m=C(Q_j,j=1,\dots,J)\vec{1}^T\vec{b}_r \qquad r=1,\dots,J$

$$Q_r=-m\,C(Q_j,j=1,\dots,J)\vec{1}^T\vec{b}_r \qquad r=1,\dots,J \qquad \text{(B.9)}$$

where $\vec{1}$ is the identity vector, $\vec{b}_r$ is the rth column of Σ_U. Since every element of $\vec{1}^T\vec{b}_r$ has the same value, the optimal solution falls in the range of values where $Q_r = Q\ \forall r$.

Note: Leibnitz's rule for differentiating integrals (Abramowitz and Stegun, 1972; Silver et al., 1998)

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)dx = \int_{a(z)}^{b(z)} \frac{\partial f(x,z)}{\partial z}dx + f(b(z),z)\frac{\partial b(z)}{\partial z} - f(a(z),z)\frac{\partial a(z)}{\partial z}$$

where *a(z)*, *b(z)* are a function of *z* and *f(x,z)* is a function of *x* and *z*.

Note: Multiplication rule for differentiation (Abramowitz and Stegun, 1972)

$$\frac{\partial f_1(z) f_2(z)}{\partial z} = f_1(z)\frac{\partial f_2(z)}{\partial z} + f_2(z)\frac{\partial f_1(z)}{\partial z}$$

where *a(z)*, *b(z)* are a function of *z* and *f(x,z)* is a function of *x* and *z*.

Appendix C: Gradient Estimation Method

This appendix explains how to estimate $\dfrac{\partial a_{i,t}}{\partial Y_{j,t-L_j}}$. From (4.17), $a^{'*}_{i,t} = a^{*}_{i,t} + \Delta a_{i,t}$.

Furthermore, both paths will have the same initial system state at period *t*, X_t. The perturbation generation effect of $\Delta Y_{j,t-L_j}$ on the allocation decision only takes effect after the order arrival at period *t*. We approximate the optimal objective function of the perturbed path by second-order Taylor's expansion series,

$$AC_{t+l}(\vec{a}^{'*}_t \mid \mathrm{X}^{'}_t, \vec{Y}^{'}) = AC_{t+l}(\vec{a}^{*}_t \mid \mathrm{X}_t, \vec{Y}\ ) + \sum_i (a^{'*}_{i,t} - a^{*}_{i,t}) \nabla AC_{t+l}(\vec{a}^{*}_t \mid \mathrm{X}_t, \vec{Y}\ ) +$$

$$\frac{1}{2}\sum_i (a^{'*}_{i,t} - a^{*}_{i,t})^2 \nabla^2 AC_{t+l}(\vec{a}^{*}_t \mid \mathrm{X}_t, \vec{Y}\ ) + o(a^{'*}_{i,t} - a^{*}_{i,t}) \qquad \text{(C.1)}$$

where $o(\Delta a_{i,t})$ stands for any $f(\Delta a_{i,t})$ function that is such that $\lim\limits_{\Delta a_{i,t} \to 0} \dfrac{f(\Delta a_{i,t})}{\Delta a_{i,t}} = 0$.

The gradient of the objective function (C.1) with respect to the allocation quantity $a^{'}_{i,t}$ can be estimated as

$$\nabla AC_{t+l}(\vec{a}^{'*}_t \mid \mathrm{X}^{'}_t, \vec{Y}^{'})$$

$$= \frac{\partial}{\partial a^{'}_{i,t}} [\, AC_{t+l}(\vec{a}^{*}_t \mid \mathrm{X}_t, \vec{Y}\ ) + \sum_i (a^{'*}_{i,t} - a^{*}_{i,t}) \nabla AC_{t+l}(\vec{a}^{*}_t \mid \mathrm{X}_t, \vec{Y}\ ) +$$

$$\frac{1}{2}\sum_i (a^{'*}_{i,t} - a^{*}_{i,t})^2 \nabla^2 AC_{t+l}(\vec{a}^{*}_t \mid \mathrm{X}_t, \vec{Y}\ ) + o(a^{'*}_{i,t} - a^{*}_{i,t}) \quad]$$

$$\approx \nabla AC_{t+l}(\hat{a}^{*}_t \mid \mathrm{X}_t, \hat{Y}\ ) + (a^{'*}_{i,t} - a^{*}_{i,t}) \nabla^2 AC_{t+l}(\hat{a}^{*}_t \mid \mathrm{X}_t, \hat{Y}\ ) \ \ \forall i \qquad \text{(C.2)}$$

Since $\nabla AC_{t+l}(\vec{a}^{*}_t \mid \mathrm{X}_t, \vec{Y}\ ) = h'_i \displaystyle\int_0^{s_{i,t}+a_{i,t}} f_{D_i^{l+1}}(u_i)du_i - \left(p_i + \sum_{j \in r(i)} h_{c_j} \right) \int_{s_{i,t}+a_{i,t}}^{\infty} f_{D_i^{l+1}}(u_i)du_i$, at the stationary points, Equation (4.7) can be rewritten as

$$\nabla AC_{t+l}(\vec{a}_t^* \mid \mathrm{X}_t, \vec{Y}\ ) + \sum_j G_{ij} \lambda_{j,t}^* \ \text{-}\ \lambda_{a_i,t}^* = 0 \qquad \forall i \tag{C.3}$$

For nominal path, *and*

$$\nabla AC_{t+l}(\vec{a}_t^{'*} \mid \mathrm{X}_t^{'}, \vec{Y}^{'}) + \sum_j G_{ij} \lambda_{j,t}^{'*} \ \text{-}\ \lambda_{a_i,t}^{'*} = 0 \qquad \forall i \tag{C.4}$$

for perturbed path, respectively.

Insert (C.2) into (C.4):

$$\nabla AC_{t+l}(\vec{a}_t^* \mid \mathrm{X}_t, \vec{Y}\ ) + (a_{i,t}^{'*} - a_{i,t}^*)\nabla^2 AC_{t+l}(\vec{a}_t^* \mid \mathrm{X}_t, \vec{Y}\ ) + \sum_j G_{ij} \lambda_{j,t}^{'*} \ \text{-}\ \lambda_{a_i,t}^{'*} = 0 \qquad \forall i \tag{C.5}$$

By subtracting (C.3) from (C.5)

$$(a_{i,t}^{'*} - a_{i,t}^*)\nabla^2 AC_{t+l}(\vec{a}_t^* \mid \mathrm{X}_t, \vec{Y}\ ) + \sum_j G_{ij}\left(\lambda_{j,t}^{'*} - \lambda_{j,t}^*\right) \text{-} \left(\lambda_{a_i,t}^{'*} - \lambda_{a_i,t}^*\right) = 0 \qquad \forall i \tag{C.6}$$

By solving Equation (4.19) for all j and Equation (C.6) for all i simultaneously, the change in allocation quantity toward every product can be determined.

We illustrated the calculation by a special scenario where only component $j1$ constraint is binding at period t and all the non-negative allocation constraints are not binding, $a_{i,t}^* > 0\ \forall i$. Thus, for all the inactive constraints, the corresponding value of the dual variables or the Lagrange Multipliers are zero, $\lambda_{j,t}^* = \lambda_{j,t}^{'*} = 0$ for $j \neq j1$ and $\lambda_{a_i,t}^* = \lambda_{a_i,t}^{'*} = 0\ \ \forall i$. Equation (C.6) becomes

$$(a_{i,t}^{'*} - a_{i,t}^*)\nabla^2 AC_{t+l}(\vec{a}_t^* \mid \mathrm{X}_t, \vec{Y}\ ) + \left(\lambda_{j1,t}^{'*} - \lambda_{j1,t}^*\right) = 0 \qquad \forall i \tag{C.7}$$

Substituting $\nabla^2 AC_{t+l}(\vec{a}_t^* \mid \mathrm{X}_t, \vec{Y}\ )$ in (C.7) by (4.14),

$$(a_{i,t}^{'*} - a_{i,t}^*)\left(h'_i + p_i + \sum_j G_{ij} h_{c_j}\right) f_{D_i^l+1}(s_{i,t} + a_{i,t}^*)\ + (\lambda_{j1,t}^{'*} - \lambda_{j1,t}^*)\ = 0 \quad \forall i$$

$$(\lambda_{j1,t}^* - \lambda_{j1,t}^{'*})\ = (a_{i,t}^{'*} - a_{i,t}^*)\left(h'_i + p_i + \sum_j G_{ij} h_{c_j}\right) f_{D_i^l+1}(s_{i,t} + a_{i,t}^*) \qquad \forall i \qquad \text{(C.8)}$$

For any $i1$ and $i2$, where both products use component $j1$,

$$(\lambda_{j1,t}^* - \lambda_{j1,t}^{'*})\ = (a_{i1,t}^{'*} - a_{i1,t}^*)\left(h'_{i1} + p_{i1} + \sum_j G_{i1j} h_{c_j}\right) f_{D_i^l+1}(s_{i1,t} + a_{i1,t}^*)$$

$$(\lambda_{j1,t}^* - \lambda_{j1,t}^{'*})\ = (a_{i2,t}^{'*} - a_{i2,t}^*)\left(h'_{i2} + p_{i2} + \sum_j G_{i2j} h_{c_j}\right) f_{D_{i2}^l+1}(s_{i2,t} + a_{i2,t}^*)$$

By equating the above two equations,

$$(a_{i1,t}^{'*} - a_{i1,t}^*)\left(h'_{i1} + p_{i1} + \sum_j G_{i1j} h_{c_j}\right) f_{D_{i1}^l+1}(s_{i1,t} + a_{i1,t}^*)$$

$$= (a_{i2,t}^{'*} - a_{i2,t}^*)\left(h'_{i2} + p_{i2} + \sum_j G_{i2j} h_{c_j}\right) f_{D_{i2}^l+1}(s_{i2,t} + a_{i2,t}^*) \qquad \forall i1 \neq i2 \ \ i1, i2 \in \{i\}_{j1}$$

or

$$(a_{i2,t}^{'*} - a_{i2,t}^*)\ = \frac{(a_{i1,t}^{'*} - a_{i1,t}^*)\left(h'_{i1} + p_{i1} + \sum_j G_{i1j} h_{c_j}\right) f_{D_{i1}^l+1}(s_{i1,t} + a_{i1,t}^*)}{\left(h'_{i2} + p_{i2} + \sum_j G_{i2j} h_{c_j}\right) f_{D_{i2}^l+1}(s_{i2,t} + a_{i2,t}^*)}$$

$$\forall i1 \neq i2 \ \ i1, i2 \in \{i\}_{j1} \qquad \text{(C.9)}$$

where $\{i\}_{j1}$ is the subset of products that uses component $j1$.

By using $\Delta a_{i1,t}$ = $(a'^{*}_{i1,t} - a^{*}_{i1,t})$ as the base, the relationship between the change in the allocation quantity with $\Delta a_{i1,t}$ can be built as shown in (C.9).

Using $\Delta a_{i1,t}$ as the base and replacing other $\Delta a_{i,t}$ in (4.19) with (C.9), respectively, we developed

$$\sum_i G_{ij1} \Delta a_{i,t}$$

$$= G_{i1j1}\ (a'^{*}_{i1,t} - a^{*}_{i1,t}) + G_{i2j1} \frac{(a'^{*}_{i1,t} - a^{*}_{i1,t})\left(h'_{i1} + p_{i1} + \sum_j G_{i1j} h_{c_j}\right) f_{D_{i1}^{l+1}}(s_{i1,t} + a^{*}_{i1,t})}{\left(h'_{i2} + p_{i2} + \sum_j G_{i2j} h_{c_j}\right) f_{D_{i2}^{l+1}}(s_{i2,t} + a^{*}_{i2,t})} + \ldots$$

$$= (a'^{*}_{i1,t} - a^{*}_{i1,t})\left(h'_{i1} + p_{i1} + \sum_j G_{i1j} h_{c_j}\right) f_{D_{i1}^{l+1}}(s_{i1,t} + a^{*}_{i1,t}) \left(\sum_i \frac{G_{ij1}}{\left(h'_i + p_i + \sum_j G_{ij} h_{c_j}\right) f_{D_i^{l+1}}(s_{i,t} + a^{*}_{i,t})} \right)$$

$$= \Delta Y_{j1,t-L_j}$$

where $i1, i \in \{i\}_{j1}$

As $h'_i = h'\ \ \forall i$, $p_i = p\ \ \forall i$, $h_{c_j} = h_c\ \ \forall j$, G_{ij1} is 1 when product i uses component $j1$ and 0 otherwise, we simplified it as

$$\frac{a'^{*}_{i1,t} - a^{*}_{i1,t}}{\Delta Y_{j1,t-L_j}} = \left(\frac{\dfrac{1}{f_{D_{i1}^{l+1}}(s_{i1,t} + a^{*}_{i1,t})}}{\sum_i \dfrac{1}{f_{D_i^{l+1}}(s_{i,t} + a^{*}_{i,t})}} \right) \qquad \forall i1, i \in \{i\}_{j1}$$

Hence,

$$\frac{\partial a_{i1,t}}{\partial Y_{j1,t-L_j}} = \lim_{\Delta Y_{j1,t-L_j} \to 0} \frac{\Delta a_{i1,t}}{\Delta Y_{j1,t-L_j}} = \lim_{\Delta Y_{j1,t-L_j} \to 0} \frac{a'^{*}_{i1,t} - a^{*}_{i1,t}}{\Delta Y_{j1,t-L_j}} \approx \left(\frac{\dfrac{1}{f_{D_{1i}^{l+1}}(s_{i1,t} + a^{*}_{i1,t})}}{\sum_i \dfrac{1}{f_{D_i^{l+1}}(s_{i,t} + a^{*}_{i,t})}} \right)$$

$$\forall i1, i \in \{i\}_{j1} \qquad \text{(C.10)}$$

This solution is dependent on the system state and the product structure.

Appendix D: Simplify the Normal Demands Convolution to Standardized Format

From (4.35),

$$\int_{-\infty}^{s_{i,t}+a_{i,t}} (s_{i,t}+a_{i,t}-u_i) f_{D_i^{l+1}}(u_i)\,du_i$$

$$= \int_{-\infty}^{s_{i,t}+a_{i,t}} (s_{i,t}+a_{i,t}-u_i) \frac{1}{\sqrt{2\pi(l+1)\sigma_i^2}} e^{-\frac{1}{2}\left(\frac{u_i-\mu_i(l+1)}{\sqrt{(l+1)\sigma_i^2}}\right)^2} du_i$$

Say, $Z_{i,t} = \dfrac{s_{i,t}+a_{i,t}-\mu_i(l+1)}{\sqrt{\sigma_i^2(l+1)}}$, $\bar{u}_i = \dfrac{u_i-\mu_i(l+1)}{\sqrt{\sigma_i^2(l+1)}}$, $d\bar{u}_i = \dfrac{1}{\sqrt{\sigma_i^2(l+1)}} du_1$ and

$\psi(\bar{u}_i) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\bar{u}_i^2\right)}$ (the standard normal distribution function).

$$= \int_{-\infty}^{Z_{i,t}} (Z_{i,t}-\bar{u}_i) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\bar{u}_i^2\right)} \sqrt{\sigma_i^2(l+1)}\,d\bar{u}_i$$

$$= \sqrt{\sigma_i^2(l+1)} \int_{-\infty}^{Z_{i,t}} (Z_{i,t}-\bar{u}_i)\psi(\bar{u}_i)\,d\bar{u}_i$$

$$= \sqrt{\sigma_i^2(l+1)} \left(E\left[Z_{i,t}-\bar{u}_i\right] - \int_{Z_{i,t}}^{\infty} (Z_{i,t}-\bar{u}_i)\psi(\bar{u}_i)\,d\bar{u}_i \right)$$

$$= \sqrt{\sigma_i^2(l+1)} \left(Z_{i,t} + \int_{Z_{i,t}}^{\infty} (\bar{u}_i-Z_{i,t})\psi(\bar{u}_i)\,d\bar{u}_i \right)$$

$$= \sqrt{\sigma_i^2(l+1)} \left(Z_{i,t} + R(Z_{i,t}) \right) \tag{D.1}$$

where

$$E[Z_{i,t}-\bar{u}_i] = \left(\int_{-\infty}^{\infty} (Z_{i,t}-\bar{u}_i)\psi(\bar{u}_i)\,d\bar{u}_i \right)$$

$$= \int_{-\infty}^{Z_{i,t}} (Z_{i,t}-\bar{u}_i)\psi(\bar{u}_i)\,d\bar{u}_i + \int_{Z_{i,t}}^{\infty} (Z_{i,t}-\bar{u}_i)\psi(\bar{u}_i)\,d\bar{u}_i$$

Similarly, from (4.36)

$$\int_{s_{i,t}+a_{i,t}}^{\infty}(u_i - s_{i,t} - a_{i,t}) f_{D_i^{l+1}}(u_i) du_i$$

$$= \int_{s_{i,t}+a_{i,t}}^{\infty} (u_i - s_{i,t} - a_{i,t}) \frac{1}{\sqrt{2\pi(l+1)\sigma_i^2}} e^{-\frac{1}{2}\left(\frac{u_i - \mu_i(l+1)}{\sqrt{(l+1)\sigma_i^2}}\right)^2} du_i$$

$$= \int_{Z_{i,t}}^{\infty} (\overline{u}_i - Z_{i,t}) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\overline{u}_i^2\right)} \sqrt{\sigma_i^2 (l+1)} d\overline{u}_i$$

$$= \sqrt{\sigma_i^2 (l+1)} \left(\int_{Z_{i,t}}^{\infty} (\overline{u}_i - Z_{i,t}) \psi(\overline{u}_i) d\overline{u}_i \right)$$

$$= \sqrt{\sigma_i^2 (l+1)} \left(R(Z_{i,t}) \right) \tag{D.2}$$

The following equivalent relation is an alternative form for $R(Z_{i,t})$ that may be employed for numerical evaluation.

$$R(Z_{i,t}) = \left(\int_{Z_{i,t}}^{\infty} (\overline{u}_i - Z_{i,t}) \psi(\overline{u}_i) d\overline{u}_i \right)$$

$$= \int_{Z_{i,t}}^{\infty} (\overline{u}_i) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\overline{u}_i^2\right)} d\overline{u}_i - Z_{i,t} \int_{Z_{i,t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\overline{u}_i^2\right)} d\overline{u}_i$$

Say, $v = \dfrac{\overline{u}_i^2}{2}$ and $dv = \overline{u}_i \; d\overline{u}_i$

$$= \int_{\left(Z_{i,t}^2/2\right)}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-v} dv - Z_{i,t} \int_{Z_{i,t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\overline{u}_i^2\right)} d\overline{u}_i$$

$$= \left[\frac{1}{\sqrt{2\pi}} e^{-v} \right]_{\left(Z_{i,t}^2/2\right)}^{\infty} - Z_{i,t}\left(1 - \Phi\left(Z_{i,t}\right)\right)$$

$$= \ 0 + \frac{1}{\sqrt{2\pi}} e^{-\left(Z_{i,t}^2 / 2\right)} - Z_{i,t}\left(1 - \Phi\left(Z_{i,t}\right)\right)$$

$$= \psi\left(Z_{i,t}\right) - Z_{i,t}\left(1 - \Phi\left(Z_{i,t}\right)\right)$$

where $\Phi\left(Z_{i,t}\right)$ is the standard normal cumulative function.

As a consequence, (D.1) becomes

$$\sqrt{\sigma_i^2 (l+1)}\left(Z_{i,t} + R(Z_{i,t})\right) \qquad = \sqrt{\sigma_i^2 (l+1)}\left(Z_{i,t} + \psi\left(Z_{i,t}\right) - Z_{i,t}\left(1 - \Phi\left(Z_{i,t}\right)\right)\right)$$

$$= \sqrt{\sigma_i^2 (l+1)}\left(\psi\left(Z_{i,t}\right) + Z_{i,t}\Phi\left(Z_{i,t}\right)\right) \tag{D.3}$$

and (D.2) becomes

$$\sqrt{\sigma_i^2 (l+1)}\left(R(Z_{i,t})\right) \qquad = \sqrt{\sigma_i^2 (l+1)}\left(\psi\left(Z_{i,t}\right) - Z_{i,t}\left(1 - \Phi\left(Z_{i,t}\right)\right)\right) \tag{D.4}$$

Appendix E: Hypotheses Testing on Means of Different Allocation Policies

The performance measure of interest is the average total inventory cost over 1000 periods. The sample size n is 30. We assume that the central limit theorem applies whether or not the underlying population of the average total inventory cost is normal. We solve the hypothesis testing by following the procedures given in Montgomery and Runger (1994). We first compare the two-echelon policy and myopic allocation policy against the pure push policy, and then we compare the myopic allocation policy against the two-echelon policy.

Let μ_0, μ_1 and μ_2 denote the sample mean of total inventory cost for the pure push policy, two-echelon policy and myopic allocation policy, respectively. S_1 and S_2 are the sample standard deviations of total inventory cost. As μ_0 is calculated by solving (4.29) iteratively, sample standard deviation does not exist.

<u>a) Two-echelon policy or myopic allocation policy vs. pure push policy</u>

Tests of hypothesis on the mean, variance unknown (pp. 404-405, Montgomery and Runger, 1994).

The parameter of interest is the difference in the sample means.

The null hypothesis: $H_0 : d_1 = \mu_0 - \mu_1 = 0$ (E.1)

The alternate hypothesis: $H_1 : d_1 = \mu_0 - \mu_1 = 0$

The significance level of $\alpha = 0.05$

The test statistic: $$T_0 = \frac{d_1}{S_1 / \sqrt{n}} \qquad \text{(E.2)}$$

which follows the t distribution with n-1 degree of freedom if the null hypothesis is true.

To test the null hypothesis, the value of the test statistic t_0 in equation (E.2) is calculated, and H_0 is rejected if either

$$t_0 > t_{\alpha/2,n-1} \quad \text{or}\, t_0 < -t_{\alpha/2,n-1} \tag{E.3}$$

where $t_{\alpha/2,n-1}$ and $-t_{\alpha/2,n-1}$ are the upper and lower $100\alpha/2$ percentage points of the t distribution with n-1 degree of freedom.

To compare the myopic allocation policy against the pure push. The null hypothesis is $H_0 : d_2 = \mu_0 - \mu_2 = 0$, and the alternate hypothesis is $H_1 : d_2 = \mu_0 - \mu_2 = 0$, while the test statistic is

$$T_0 = \frac{d_2}{S_2 / \sqrt{n}} \tag{E.4}$$

Summary of the hypothesis testing is tabulated in Table E.1. Please refer to Table 4.1 for the base case setting.

Table E.1: Hypothesis testing on the means (two-echelon, myopic vs. pure push)

		Pure Push	Two-echelon		Myopic					
		μ_0	μ_1	S_1	μ_2	S_2	d_1 *	t_0 #	d_2 **	t_0 ##
L	1	372.360	369.070	6.272	359.677	6.091	3.290	2.8732	12.683	11.403
	3	422.522	406.937	7.578	385.711	6.664	15.584	11.262	36.8111	30.254
	5	463.331	436.252	8.605	406.071	7.760	27.078	17.234	57.259	40.411
	7	498.678	461.274	9.964	424.415	8.410	37.403	20.560	74.262	48.364
	9	530.336	482.913	10.575	440.048	9.142	47.422	24.561	90.287	54.090
l	1	463.331	436.252	8.605	406.071	7.760	27.078	17.234	57.259	40.411
	3	898.536	881.884	10.293	855.043	9.240	16.652	8.861	43.493	25.780
	5	1330.045	1318.573	12.980	1294.059	10.337	11.471	4.840	35.985	19.066
	7	1758.796	1750.208	13.569	1727.892	11.143	8.588	3.466	30.904	15.190
	9	2185.409	2178.199	14.738	2158.360	11.997	7.210	2.679	27.049	12.349
σ	6	357.970	341.715	4.973	323.681	5.995	16.255	17.900	34.289	31.323
	8	410.633	388.957	6.554	364.876	6.933	21.675	18.112	45.756	36.148
	10	463.331	436.252	8.605	406.071	7.760	27.078	17.234	57.259	40.411
	12	516.114	483.492	10.000	447.267	8.509	32.621	17.866	68.846	44.313
	14	569.022	530.718	11.557	488.463	9.297	38.304	18.153	80.559	47.458
H	1	463.331	436.252	8.605	406.071	7.760	27.078	17.234	57.259	40.411
	2	535.264	480.606	8.709	449.562	8.076	54.658	34.374	85.701	58.123
	3	596.261	517.067	8.996	485.498	8.214	79.194	48.217	110.763	73.851
	4	649.354	548.244	9.435	516.366	8.405	101.110	58.694	132.989	86.658
h_C	1	463.331	436.252	8.605	406.071	7.760	27.078	17.234	57.259	40.411
	2	753.826	737.716	11.075	691.571	9.027	16.109	7.966	62.254	37.772
	3	1027.583	1017.042	12.780	959.061	10.006	10.540	4.517	68.521	37.506
	4	1290.317	1282.877	13.918	1215.145	10.755	7.439	2.927	75.171	38.280
p	10	463.331	436.252	8.605	406.071	7.760	27.078	17.234	57.259	40.411
	20	519.497	482.054	10.467	445.617	8.980	37.443	19.592	73.880	45.062
	30	551.095	507.037	11.940	467.516	9.654	44.058	20.209	83.579	47.418
	40	572.810	523.957	13.159	482.383	10.093	48.85	20.333	90.427	49.070
	50	589.233	537.051	14.1338	493.632	10.438	52.181	20.221	95.600	50.162

* $d_1 = \mu_0 - \mu_1$

\# calculated from (E.2)

** $d_2 = \mu_0 - \mu_2$

\## calculated from (E.4)

The results show that the null hypothesis is rejected because t_0 calculated > $t_{0.025,29}$ = 2.3638. Both the two-echelon policy and the myopic allocation policy have significantly lower average total inventory cost than the pure push policy at 95% confidence level.

b) Myopic allocation policy vs. two-echelon policy

In this section, we first present the procedures to test the equality of two variances (S_1^2 and S_2^2) and then conduct the pooled-t test on the means (μ_1 and μ_2)

Tests of hypothesis on the variance (a large-sample test procedure – p. 434, Montgomery and Runger, 1994). The parameters of interest are S_1^2 and S_2^2. This test is based on the assumption that the sample standard deviation S_1 and S_2 have approximate normal distributions with mean σ_1 and σ_2, respectively. The sample size, $n_1 = n_2 = n = 30$.

The null hypothesis: $H_0 : \sigma_1^2 = \sigma_2^2$ (E.5)

The alternate hypothesis: $H_1 : \sigma_1^2 \neq \sigma_2^2$

The significance level of $\alpha = 0.05$

The test statistic $f_0 = \dfrac{S_1^2}{S_2^2}$ (E.6)

This test statistic has an F distribution with n – 1 numerator degrees of freedom and n - 1 denominator degree of freedom.

The null hypothesis is rejected if the value of the test statistic $f_0 > f_{0.025,29,29}$ = 2.10 or $f_0 < f_{0.975,29,29} = 1/f_{0.025,29,29}$ = 0.475. Based on the hypothesis testing which is summarized in Table E.2, we are unable to reject the null hypothesis $H_0 : \sigma_1^2 = \sigma_2^2$ at the $\alpha = 0.05$ level of significance. That is, there is no strong evidence indicating that the variances are different.

We now consider tests of hypothesis on the equality of the means $\mu_1 = \mu_2$ where the variances σ_1 and σ_2 are unknown but equal. The two-sample *t*-test given in this section is often called the pooled *t* test. (pp. 410-412, Montgomery and Runger, 1994)

The parameter of interest is the difference in the mean. $d_3 = \mu_1 - \mu_2$

The null hypothesis: $H_0 : d_3 = \mu_1 - \mu_2 = 0$ (E.7)

The alternate hypothesis: $H_1 : d_3 = \mu_1 - \mu_2 \neq 0$

The significance level of $\alpha = 0.05$

The test statistic: $$T_0 = \frac{d_3}{S_p \sqrt{2/n}}$$ (E.8)

where S_p is the pooled estimator of the common standard deviation. $S_p^2 = \frac{(n-1)\left(S_1^2 + S_2^2\right)}{2n-2}$. This test statistic follows the *t* distribution with *n*-1 degree of freedom. The null hypothesis is rejected if the value of the test statistic $t_0 > t_{\alpha/2,2n-2}$ or $t_0 < -t_{\alpha/2,2n-2}$, where $t_{\alpha/2,2n-2}$ and $-t_{\alpha/2,2n-2}$ are the upper and lower 100 α /2 percentage points of the *t* distribution. The acceptance interval is -2.301 = $-t_{0.025,58} < t_o < t_{0.025,58} = 2.301$.

Table E.2: Hypothesis testing (myopic vs. two-echelon)

		$d_3 = \mu_1 - \mu_2$	S_p^2	Test of Variance f_0 *	Test of Mean t_0 **
L	1	9.392	38.224	1.060	5.883
	3	21.226	50.926	1.293	11.519
	5	30.180	67.142	1.230	14.265
	7	36.859	85.007	1.404	15.483
	9	42.864	97.710	1.338	16.795
l	1	30.180	67.142	1.230	14.265
	3	26.840	95.669	1.241	10.628
	5	24.513	137.688	1.577	8.091
	7	22.316	154.146	1.483	6.961
	9	19.839	180.578	1.509	5.717
σ	6	18.033	30.343	0.688	12.679
	8	24.081	45.516	0.894	13.824
	10	30.180	67.142	1.230	14.265
	12	36.224	86.211	1.381	15.110
	14	42.254	110.006	1.545	15.603
H	1	30.180	67.142	1.230	14.265
	2	31.043	70.538	1.163	14.315
	3	31.568	74.206	1.199	14.193
	4	31.877	79.839	1.260	13.817
h_C	1	30.180	67.141	1.230	14.265
	2	46.145	102.085	1.505	17.689
	3	57.980	131.729	1.631	19.565
	4	67.732	154.698	1.675	21.091
p	10	30.180	67.142	1.230	14.265
	20	36.436	95.107	1.359	14.470
	30	39.521	117.894	1.530	14.097
	40	41.574	137.531	1.700	13.730
	50	43.419	154.365	1.834	13.535

* calculated from (E.6) ** calculated from (E.8)

On the other hand, since $t_0 > t_{0.025,58} = 2.301$, we conclude that the myopic allocation policy produces, on average, a lower total inventory cost than does the two-echelon policy. The t_0 value is considerably higher, so the test statistic is well into the critical region.

Appendix F: Hypothesis Testing on Means of Eppen and Schrage's Heuristic and Myopic Allocation Policy

For this hypothesis testing, we followed the process mentioned in Section (b) Appendix E. Firstly, we tested the equality of two variances (S_1^2 and S_2^2) and then conducted the pooled-t test on the means (μ_1 and μ_2). Results are summarized in Table F.1. Tests of hypothesis on the variance are based on a large-sample test procedure (p. 434, Montgomery and Runger, 1994). While tests of hypothesis on the equality of the means $\mu_1 = \mu_2$ where the variances σ_1 and σ_2 are unknown but equal are based on the two-sample pooled *t*-test (pp. 410-412, Montgomery and Runger, 1994).

Table F.1: Hypothesis testing on Eppen and Schrage method (sample 1) vs. proposed method (sample 2)

				Test of Variance	Test of Mean
p	h'	$d_3 = \mu_1 - \mu_2$	S_p^2	f_0*	t_0**
10	1	-0.647	43.50	1.399	-0.380
	10	47.453	55.47	0.745	24.675
	30	139.251	114.32	0.923	50.440
	50	193.589	53.52	1.178	102.482
30	1	11.262	24.365	1.779	8.837
	10	47.024	58.379	1.141	23.836
	30	147.287	136.374	1.858	48.848
	50	335.891	219.258	2.019	87.855
50	1	7.185	49.987	0.753	3.936
	10	39.833	91.656	1.098	16.114
	30	207.583	239.029	1.742	52.001
	50	377.644	383.275	1.746	74.709

* calculated from (E.6) ** calculated from (E.8)

From Table F.1, all f_0's are within $0.475 = f_{0.975,29,29} = 1/f_{0.025,29,29} < f_0 < f_{0.025,29,29} = 2.10$. The null hypothesis (E.5) cannot be rejected and it can be concluded that there is no strong evidence that the variances S_1^2 and S_2^2 are different.

On the other hand, except for the case p = 10 and h' = 1, all $t_0 > t_{0.025,58} = 2.301$. We have concluded that the myopic allocation policy produces, on average, a lower total inventory cost than that of Eppen and Schrage's heuristic. The t_0 value is considerably higher, so the test statistic is well into the critical region.

Appendix G: Conditional Mean and Variance of Multi-Period Forecast Demands

Given that the demand is stationary and has an auto-regressive process of order P, and the coefficent of auto-correlation matrix, $\vec{P}_i$, the conditional mean $\mu_i(k \mid \vec{d})$ as in (5.10) and the conditional variance $\sigma_i^2(k \mid \vec{d})$ as in (5.11) can be computed from the time series forecasting method proposed by Box et al. (1994). The auto-correlation matrix, $P_{i,p}$ is sysmetric with identity element (one) on the diagonal. The main results are discussed as follows.

Suppose the current time is t, and given the past demand information, i.e. $\vec{d}$ = ($d_{i,t\text{-}1}$, $d_{i,t\text{-}2}$, …, $d_{i,t\text{-}p}$), we would like to predict the current demand of product i at period t, denoted by $D_{i,t}$, which has not been realized when the order decision or the allocation decision are made. If we define $z_{i,t} = D_{i,t} - \mu_i$, where μ_i is the mean demand of product i, then the AR(P) process, where the deviation $z_{i,t}$ is regressed on past P observations plus an added white noise $\varepsilon_{i,t}$, is defined as follows

$$z_{i,t} = \phi_{i,1} z_{i,t-1} + \phi_{i,2} z_{i,t-2} + \cdots + \phi_{i,\mathrm{P}} z_{i,t-\mathrm{P}} + \varepsilon_{i,t} \tag{G.1}$$

where $\phi_{i,j}$, j = 1,...P, are the weight parameters and $\varepsilon_{i,t}$ is the white noise from the auto-regressive process of product i at period t. The random variables, $\varepsilon_{i,s}$'s, are uncorrelated and have zero means and homogeneous variances, σ'^2_i .

By multiplying throughout in (G.1) by $z_{i,t-k}$ to obtain

$$z_{i,t-k} z_{i,t} = \phi_{i,1} z_{i,t-k} z_{i,t-1} + \phi_{i,2} z_{i,t-k} z_{i,t-2} + \cdots + \phi_{i,\mathrm{P}} z_{i,t-k} z_{i,t-\mathrm{P}} + z_{i,t-k} \varepsilon_{i,t} \tag{G.2}$$

On taking expected values in (G.2), the equation becomes

$$\gamma_{i,k} = \phi_{i,1}\gamma_{i,k-1} + \phi_{i,2}\gamma_{i,k-2} + \cdots + \phi_{i,\mathrm{P}}\gamma_{i,k-\mathrm{P}} \tag{G.3}$$

where $\gamma_{i,k}$ = $[z_{i,t-k}z_{i,t}] = E[(z_{i,t} - E[z_{i,t}])(z_{i,t-k} - E[z_{i,t-k}])]$ is the autocovariance of product i at lag k and $\gamma_{i,0}$ = σ_i^2. For a stationary process the variance.?? Note that the expectation E $[z_{i,t-k}\varepsilon_{i,t}]$ vanishes when k > 0, since $z_{i,t-k}$ contains noise up to $\varepsilon_{i,t-k}$ only.

Dividing throughout (G.3) by $\gamma_{i,0}$

$$\rho_{i,k} = \phi_{i,1}\rho_{i,k-1} + \phi_{i,2}\rho_{i,k-2} + \cdots + \phi_{i,\mathrm{P}}\rho_{i,k-\mathrm{P}} \tag{G.4}$$

where $\rho_{i,k}$ = $\dfrac{E[(z_{i,t} - E[z_{i,t}])(z_{i,t-k} - E[z_{i,t-k}])]}{\sqrt{E[(z_{i,t} - E[z_{i,t}])]^2 E[(z_{i,t-k} - E[z_{i,t-k}])^2]}}$ is the autocorrelation of product i at lag k.

If we substitute k = 1,2,3,… P in (G.4), we can obtain a set of linear equations which can be represented by

$$\vec{\rho}_i = \vec{P}_i\,\vec{\phi}_i \tag{G.5}$$

where $\vec{\phi}_i = \begin{bmatrix} \phi_{i,1} \\ \phi_{i,2} \\ \vdots \\ \phi_{i,\mathrm{P}} \end{bmatrix}$; $\vec{\rho}_i = \begin{bmatrix} \rho_{i,1} \\ \rho_{i,2} \\ \vdots \\ \rho_{i,\mathrm{P}} \end{bmatrix}$ and $\vec{P}_i = \begin{bmatrix} 1 & \rho_{i,1} & \rho_{i,2} & \cdots & \rho_{i,\mathrm{P}-1} \\ \rho_{i,1} & 1 & \rho_{i,1} & \cdots & \rho_{i,\mathrm{P}-2} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \rho_{i,\mathrm{P}-1} & \rho_{i,\mathrm{P}-2} & \rho_{i,\mathrm{P}-3} & \cdots & 1 \end{bmatrix}$.

By multiplying throughout in (G.5) by $\vec{P}_i^{-1}$ to obtain

$$\vec{\phi}_i = \vec{P}_i^{-1}\vec{\rho}_i \tag{G.6}$$

This relationship between the weight parameters and the correlation coefficients is also best known as the Yule-Walker estimator $\vec{\phi}_i$ and $\vec{\sigma}_i^2$ of ϕ_i and σ_i^2 (Walker, 1931; Yule, 1927).

On taking expected values in (G.4) when $k = 0$,

$$\gamma_{i,0} = \phi_{i,1}\gamma_{i,-1} + \phi_{i,2}\gamma_{i,-2} + \cdots + \phi_{i,\mathrm{P}}\gamma_{i,-\mathrm{P}} + \sigma'^2_i \qquad \text{(G.7)}$$

where $E[z_{i,t}z_{i,t}] = \gamma_{i,0} = \sigma_i^2$ and $E[z_{i,t}\varepsilon_{i,t}] = E[\varepsilon_{i,t}^2] = \sigma'^2_i$.

On dividing throughout by $\gamma_{i,0} = \sigma_i^2$ and substituting $\gamma_{i,-k} = \gamma_{i,k}$, the variance σ'^2_i can be expressed as

$$1 = \phi_{i,1}\rho_{i,1} + \phi_{i,2}\rho_{i,2} + \cdots + \phi_{i,\mathrm{P}}\rho_{i,\mathrm{P}} + \frac{\sigma'^2_i}{\sigma_i^2}$$

$$\sigma'^2_i = \left(1 - \phi_{i,1}\rho_{i,1} - \phi_{i,2}\rho_{i,2} - \cdots - \phi_{i,\mathrm{P}}\rho_{i,\mathrm{P}}\right)\sigma_i^2 \qquad \text{(G.8)}$$

Alternatively, in general terms,

$$\sigma'^2_i = \sigma_i^2\left(1 - \vec{\rho}_i{}'\vec{P}_i^{-1}\vec{\rho}_i\right) \qquad \text{(G.9)}$$

It can be shown that E[$z_{i,t}$] + μ_i (where $z_{i,t}$ given in (G.1)) and σ'^2_i are, respectively, the conditional mean and the conditional variance of $D_{i,t}$ given $\vec{d}$. Therefore, the conditional distribution of $D_{i,t}$ given $\vec{d}$ is normally with mean E[$z_{i,t}$] + μ_i and variance σ'^2_i. Note that this derivation estimates the parameters of the conditional distibution of the demand at any one period. However, it is necessary to estimate the conditional mean and the conditional variance of the demand over a duration of time.

Suppose we would like to find the conditional mean and the conditional variance of $\sum_{k=0}^{l} D_{i,t+k}$ given $\vec{d}$. It can be shown that

$$E\left[\sum_{k=0}^{l} D_{i,t+k} \middle| \vec{d}\right] = (l+1)\mu_i + \sum_{k=0}^{l} E[z_{i,t+k}] \tag{G.10}$$

where the values of $z_{i,t+k}$, $k = 0,1,...,l$ can be obtained recursively from (G.1). Note that the weights, $\phi_{i,k}$, $k = 1,...,\mathrm{P}$ are fixed and they are obtained from (G.7). Furthermore, given X_t, the conditional variance of $\sum_{k=0}^{l} D_{i,t+k}$ is

$$\begin{aligned} Var\left[\sum_{k=0}^{l} D_{i,t+k} \middle| \vec{d}\right] &= Var\left[\sum_{k=0}^{l} z_{i,t+k} \middle| \vec{d}\right] \\ &= \sum_{m=0}^{l}\sum_{n=0}^{l} \gamma_{i,|m-n|} \end{aligned} \tag{G.11}$$

where $|m-n|$ is the modulus of m-n since $\gamma_{i,k} = \gamma_{i,-k}$.

For variance calculation, we needed to introduce another equivalent form of linear process because it is hard to estiamte $\gamma_{i,k}$ using (G.3) due to interdependency. By substituting $z_{i,t-1} = \phi_{i,1} z_{i,t-2} + \phi_{i,2} z_{i,t-3} + \cdots + \phi_{i,\mathrm{P}} z_{i,t-\mathrm{P}-1} + \varepsilon_{i,t-1}$ in (G.1),

$$z_{i,t} = \phi_{i,1}\left(\phi_{i,1} z_{i,t-2} + \phi_{i,2} z_{i,t-3} + \cdots + \phi_{i,\mathrm{P}} z_{i,t-\mathrm{P}-1} + \varepsilon_{i,t-1}\right) + \phi_{i,2} z_{i,t-2} + \cdots + \phi_{i,\mathrm{P}} z_{i,t-\mathrm{P}} + \varepsilon_{i,t}$$

$$\begin{aligned} z_{i,t} &= (\phi_{i,1}^2 + \phi_{i,2}) z_{i,t-2} + (\phi_{i,1}\phi_{i,2} + \phi_{i,3}) z_{i,t-3} + ...(\phi_{i,1}\phi_{i,\mathrm{P}-1} + \phi_{i,\mathrm{P}}) z_{i,t-\mathrm{P}} \\ &\quad + (\phi_{i,1}\phi_{i,\mathrm{P}}) z_{i,t-\mathrm{P}-1} + \varepsilon_{i,t} + \phi_{i,1}\varepsilon_{i,t-1} \end{aligned}$$

Similarly, by substituting $z_{i,t-2} = \phi_{i,1} z_{i,t-3} + \phi_{i,2} z_{i,t-4} + \cdots + \phi_{i,\mathrm{P}} z_{i,t-\mathrm{P}-2} + \varepsilon_{i,t-2}$

$$\begin{aligned} z_{i,t} &= (\phi_{i,1}^2 + \phi_{i,2})\left(\phi_{i,1} z_{i,t-3} + \phi_{i,2} z_{i,t-4} + \cdots + \phi_{i,\mathrm{P}} z_{i,t-\mathrm{P}-2} + \varepsilon_{i,t-2}\right) \\ &\quad + (\phi_{i,1}\phi_{i,2} + \phi_{i,3}) z_{i,t-3} + ...(\phi_{i,1}\phi_{i,\mathrm{P}-1} + \phi_{i,\mathrm{P}}) z_{i,t-\mathrm{P}} + (\phi_{i,1}\phi_{i,\mathrm{P}}) z_{i,t-\mathrm{P}-1} + \varepsilon_{i,t} + \phi_{i,1}\varepsilon_{i,t-1} \end{aligned}$$

$$z_{i,t} = (\phi_{i,1}^3 + \phi_{i,1}\phi_{i,2} + \phi_{i,2}) z_{i,t-3} + ... + .((\phi_{i,1}^2\phi_{i,\mathrm{P}-2} + \phi_{i,2}\phi_{i,\mathrm{P}-2}) + \phi_{i,1}\phi_{i,\mathrm{P}-1} + \phi_{i,\mathrm{P}}) z_{i,t-\mathrm{P}} +$$

$$(\phi_{i,1}^2\phi_{i,\mathrm{P}-1} + \phi_{i,2}\phi_{i,\mathrm{P}-1} + \phi_{i,1}\phi_{i,\mathrm{P}})z_{i,t-\mathrm{P}-1} + (\phi_{i,1}^2 + \phi_{i,2})\phi_{i,\mathrm{P}}z_{i,t-\mathrm{P}-2}$$

$$+\varepsilon_{i,t} + \phi_{i,1}\varepsilon_{i,t-1} + (\phi_{i,1}^2 + \phi_{i,2})\varepsilon_{i,t-2}$$

This equation will eventually be an infinite weighted sum of present and past values of the white noise process $\varepsilon_{i,t}$ (Koopmans, 1974)

$$z_{i,t} = \varepsilon_{i,t} + \psi_{i,1}\varepsilon_{i,t-1} + \psi_{i,2}\varepsilon_{i,t-2} + \psi_{i,3}\varepsilon_{i,t-3} + \dots\dots\dots$$

$$= \sum_{k=0}^{\infty}\psi_{i,k}\varepsilon_{i,t-k} \tag{G.12}$$

where $\psi_{i,k}$ is the weighted parameter.

The value of $\psi_{i,k}$ can be obtained recursively from

$$\psi_{i,k} = \sum_{m=1}^{k}\psi_{i,k-m}\phi_{i,m} \tag{G.13}$$

where $\psi_{i,0} = 1$.

Therefore, the autocovariance of product *i* at lag *k* of (G.12)

$$\gamma_{i,k} = E[z_{i,t}z_{i,t+k}]$$

$$= E[(\varepsilon_{i,t} + \psi_{i,1}\varepsilon_{i,t-1} + \psi_{i,2}\varepsilon_{i,t-2} + \dots)(\varepsilon_{i,t+k} + \psi_{i,1}\varepsilon_{i,t+k-1} + \psi_{i,2}\varepsilon_{i,t+k-2} + \dots)]$$

$$= E\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\psi_{i,m}\psi_{i,n}\varepsilon_{i,t-m}\varepsilon_{i,t+k-n}\right]$$

$$= \sigma'^2_i\sum_{m=0}^{\infty}\psi_{i,m}\psi_{i,m+k} \tag{G.14}$$

As the white noise, $\varepsilon_{i,s}$'s, are uncorrelated and have zero means and homogeneous variances, σ'^2_i, from (G.14), by setting $k = 0$,

$$\gamma_{i,0} = \sigma_i^2 = \sigma'^2_i \sum_{m=0}^{\infty} \psi_{i,m}^2 \tag{G.15}$$

It follows that the stationary condition necessary for the coefficient $\psi_{i,k}$ to be absolutely summable is $\sum_{k=0}^{\infty} \left| \psi_{i,k} \right| < \infty$. This implies that the series in (G.15) converges and hence guarantees that the series has a finite variance (Wold 1954). In other words, any non-zero mean purely non-deterministic stationary process, $z_{i,t}$, possesses a linear representation as in (G.12) with $\sum_{k=0}^{\infty} \psi_{i,k}^2 < \infty$.

$$\begin{aligned} Var\left[\sum_{k=0}^{l} D_{i,t+k} \middle| \vec{d} \right] &= \sum_{m=0}^{l} \sum_{n=0}^{l} \gamma_{i,|m-n|} \\ &= (l+1)\gamma_{i,0} + 2(l+1-n)\sum_{n=1}^{l} \gamma_{i,n} \\ &= \sigma'^2_i \left[(l+1)\sum_{m=0}^{\infty} \psi_{i,m}^2 + 2(l+1-n)\sum_{m=0}^{\infty} \psi_{i,m}\psi_{i,m+n} \right] \end{aligned} \tag{G.16}$$

Therefore, the conditional mean and conditional variance of $\sum_{k=0}^{l} D_{i,t+k}$ are

$$\mu_i(l+1 \mid \vec{d}) = (l+1)\mu_i + \Lambda(\vec{d},i,l+1) = (l+1)\mu_i + \sum_{k=0}^{l} E[z_{i,t+k}] \tag{G.17}$$

where $\Lambda(\vec{d},i,l+1) = \sum_{k=0}^{l} E[z_{i,t+k}]$ (G.18)

and,

$$\sigma_i^2(l+1 \mid \vec{d}) = \sigma'^2_i\, \Gamma(\vec{d},i,l+1) = \sigma'^2_i \left[(l+1)\sum_{m=0}^{\infty} \psi_{i,m}^2 + 2(l+1-n)\sum_{m=0}^{\infty} \psi_{i,m}\psi_{i,m+n} \right] \tag{G.19}$$

respectively. Note that the conditional mean of the demand is always changing at every period. Hence, the demand parameters, the conditional mean $\mu_i(l+1 \mid \vec{d})$ and the conditional variance $\sigma_i^2(l+1\,|\,\vec{d})$, are updated at every period. The allocation decision and the procurement decision are accordingly made based on the updated parameters.

Appendix H: Hypothesis Testing on Means of Different Procurement Policies

<u>a) Dynamic Level with Sharing vs. Dynamic Level without Sharing</u>

For this hypothesis testing, we followed the process mentioned in Section (b) Appendix E. Firstly, we tested the equality of two variances (S_1^2 and S_2^2) which show that the hypothesis cannot be rejected and then conducted the pooled-t test on the means (μ_1 and μ_2). Results are summarized in Table H.1.

From Table H.1, all f_0's are within 0.475 = $f_{0.975,29,29} = 1/f_{0.025,29,29} < f_0 < f_{0.025,29,29}$ = 2.10. The null hypothesis (E.5) cannot be rejected and it can be concluded that there is no strong evidence that the variances S_1^2 and S_2^2 are different.

On the other hand, all t_0 > $t_{0.025,58}$ = 2.301. We can conclude that the Dynamic Level with Sharing, on average, has a lower total cost than the Dynamic Level without Sharing. The t_0 value is considerably higher, so the test statistic is well into the critical region.

Table H.1: Hypothesis testing on dynamic level without sharing (sample 1) and dynamic level with sharing (sample 2)

Order of Auto-regressive				Test of Variance	Test of Mean
P	ρ	$d_3 = \mu_1 - \mu_2$	S_p^2	f_0 *	t_0 **
1	0.9	77.798	245.696	0.794	27.185
	0.7	104.235	268.003	0.824	34.874
	0.5	106.294	237.867	0.843	37.749
	0.3	95.004	288.986	0.592	30.610
	0.1	71.170	245.255	0.928	24.891
	0	59.193	212.469	0.973	22.243
5	0.9	28.012	211.492	0.972	10.550
	0.7	64.677	297.967	0.494	20.522
	0.5	75.970	195.408	0.900	29.767
	0.3	79.028	200.763	0.911	30.549
	0.1	68.419	176.842	0.917	28.180
	0	59.193	212.469	0.973	22.243
10	0.9	19.538	250.786	0.852	6.757
	0.7	53.478	240.203	1.014	18.899
	0.5	64.044	277.001	0.778	21.076
	0.3	73.994	232.700	1.048	26.568
	0.1	72.658	298.103	0.805	23.049
	0	59.193	212.469	0.973	22.243
14	0.9	18.607	216.423	0.787	6.927
	0.7	34.865	290.581	0.719	11.203
	0.5	43.006	221.903	1.277	15.813
	0.3	70.801	283.770	0.791	23.021
	0.1	71.608	289.822	0.807	23.039
	0	59.193	212.469	0.973	22.243

* calculated from (E.6) ** calculated from (E.8)

b) Dynamic Level with Sharing vs. Constant Level with Sharing

Firstly, we tested the equality of two variances (S_1^2 and S_2^2) from the two samples, which illustrates that we have strong evidence, at 95% confidence level, to reject the hypothesis testing that the two samples do not have equal variance because f_0 < $f_{0.975,29,29} = 1/f_{0.025,29,29}$ =0.475. The result is summarized in Table H.2.

We consider tests of hypothesis on the equality of the means $\mu_1 = \mu_2$ where the variances σ_1 and σ_2 are unknown and unequal. The test statistic is used for this pooled *t* test when $\sigma_1 \neq \sigma_2$ (pp. 410-412, Montgomery and Runger, 1994).

The parameter of interest is the difference in the mean $d_3 = \mu_1 - \mu_2$

The null hypothesis: $H_0 : d_3 = \mu_1 - \mu_2 = 0$ (H.1)

The alternate hypothesis: $H_1 : d_3 = \mu_1 - \mu_2 \neq 0$

The significance level of $\alpha = 0.05$

The test statistic
$$T_0 = \frac{d_3}{\sqrt{\frac{S_1^2}{n} + \frac{S_2^2}{n}}} \qquad \text{(H.2)}$$

where *n* is the sample size, *n* = 30. This test statistic follows the *t* distribution with *v* degree of freedom which is given by

$$v = \frac{\left(\frac{S_1^2}{n} + \frac{S_2^2}{n}\right)^2}{\frac{\left(S_1^2/n\right)^2 + \left(S_2^2/n\right)^2}{n+1}} - 2 \qquad \text{(H.3)}$$

The null hypothesis is rejected if the value of the test statistic $t_0 > t_{\alpha/2,v}$ or $t_0 < -t_{\alpha/2,v}$, where $t_{\alpha/2,\nu}$ and $-t_{\alpha/2,\nu}$ are the upper and lower $100\alpha/2$ percentage points of the t distribution.

Table H.2: Hypothesis testing on constant level with sharing (sample 1) and dynamic level with sharing (sample 2)

Order of Auto-regressive			Test of Variance	Test of Mean	Degree of Freedom	t Critical
P	ρ	$d_3 = \mu_1 - \mu_2$	f_0 *	t_0 **	ν #	
1	0.9	108.675	0.401	22.993	50.426	2.311
	0.7	21.522	0.263	4.595	44.240	2.321
	0.5	11.647	0.218	2.689	41.886	2.327
	0.3	10.542	0.225	2.192	42.297	2.325
	0.1	-1.145	0.212	-0.261	41.592	2.327
	0	-6.497	0.216	-1.592	41.773	2.327
5	0.9	295.257	0.378	68.104	49.497	2.312
	0.7	171.577	0.257	34.624	43.958	2.323
	0.5	96.387	0.390	23.000	49.985	2.312
	0.3	39.356	0.368	9.340	49.107	2.312
	0.1	2.051	0.397	0.513	50.267	2.311
	0	-6.483	0.318	-1.525	46.893	2.317
10	0.9	328.914	0.224	73.763	42.235	2.325
	0.7	215.136	0.308	47.815	46.461	2.317
	0.5	136.664	0.210	29.296	41.471	2.327
	0.3	74.547	0.313	16.816	46.656	2.317
	0.1	12.916	0.178	2.707	39.682	2.331
	0	-6.483	0.275	-1.551	44.852	2.321
14	0.9	328.109	0.271	77.663	44.634	2.321
	0.7	220.198	0.310	44.244	46.551	2.317
	0.5	134.661	0.305	31.318	46.290	2.317
	0.3	85.944	0.176	18.465	39.606	2.331
	0.1	17.486	0.183	3.707	39.994	2.331
	0	-6.483	0.275	-1.551	44.852	2.321

* calculated from (E.6) ** calculated from (H.2)

\# calculated from (H.3)

Table H.2 exhibits that when ρ is small (0 or 0.1), there is no difference in using Dynamic Level with Sharing or Constant Level with Sharing because both policies' performance is statistically insignificant. When the demand becomes more positively correlated, ρ is higher, Dynamic Level with Sharing becomes a better choice.